Dade's Ordinary Conjecture for the Finite Special Unitary Groups: Part II

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Abstract

This report is a continuation of the proof of Dade's Ordinary Conjecture (DOC) for the finite special unitary groups. Several reductions of the main alternating sum were completed in Part I [1] resulting in an important reformulation of the main theorem. The alternating sum in this theorem was immediately decomposed into two sub alternating sums. The aim of this paper is to prove the second of these sub alternating sums, completing the proof of DOC for the finite special unitary groups in the defining characteristics.

Let us recall the context from [1]. Let G be a finite group. An ordinary character of G is the character of a representation of G over a field of characteristic 0. In the p-modular representation theory of G,where p is a prime dividing the order of G, the ordinary irreducible characters of G are divided into disjoint sets called p-blocks which reflect the decomposition of the group algebra of G over a field of characteristic p into indecomposable two-sided ideals. An important problem is to classify the p-blocks, and a first step is to count the number of ordinary characters in a block.

The aim of DOC is to prove an alternating sum of the form

$$\sum_{C/G} (-1)^{|C|} k(N_G(C), B, d) = 0, \qquad \forall d \geq 0$$

which counts the number of characters in B in terms of corresponding numbers in subgroups of G which are normalizers of chains of certain p-subgroups of G.

This has been shown for p-blocks, p dividing q, for $\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$ and $\mathrm{U}_n(q)$. We prove DOC for $\mathrm{SU}_n(q)$. The main difficulties involved arise because the structure of the unitary groups is more complicated than that of the linear groups. In particular the cancellations in the alternating sum in the unitary case are very different from the cancellations that occur in the general linear case. A key result is that a version of Ku's parametrization of characters for $\mathrm{U}_n(q)$ survives restriction to $\mathrm{SU}_n(q)$.

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1 Introduction

We begin by restating the main results from Part I. While some definitions are also restated for clarity, the reader is encouraged to revisit Part I [1].

1.1 Statement of Dade's Ordinary Conjecture for $\mathrm{U}_{\mathbf{n}}(\mathbf{q})$

Recall the set up in Section 5 of Part I. Let $J \subset I = [m]$, an index set for the distinguished generators of the Weyl group of type B_m. Let P_J be the standard parabolic subgroup of $\mathrm{U}_n(q)$ corresponding to J. Then $P_J = \bigcap_{j \in J} P_j$ where P_j is the maximal parabolic subgroup corresponding to j. We have the usual Levi decomposition of $P_J = L_J U_J$ where L_J is a Levi subgroup and U_J is the unipotent radical of P_J.

Recall that $k_d(P_J, \rho, \det, j)$ denotes the number of irreducible ordinary characters χ of the parabolic subgroup P_J with q-height d and lying over the central character ρ such that the restriction of χ to the kernel of the map ϕ has j' irreducible components, where j divides j'. Moreover the q-height of χ is d if $q^d \| \chi(1)$.

Theorem 1.1 (Main) *Let $Z = Z(\mathrm{U}_n(q))$ and $\{P_J | J \subseteq I\}$ the set of standard parabolic subgroups in $\mathrm{U}_n(q)$. For any $\rho \in \mathrm{Irr}(Z)$, any positive integer j, and all nonnegative integers d we have*

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho, \det, j) = \begin{cases} \beta((n), a_\rho), & \text{if } d = \binom{n}{2} \text{ and } j=1; \\ 0, & \text{otherwise.} \end{cases}$$

In order to prove Theorem 1.1 we first broke the left hand side into two sub-sums. We differentiated between characters χ counted by $k_d(P_J, \rho, \det, j)$ for which $\ker \chi$ contains U_J or not.

Definition 1.2 *Let $k_d^0(P_J, U_J, \rho, \det, j)$ be the number of characters counted by $k_d(P_J, \rho, \det, j)$ which contain U_J in their kernel and let $k_d^1(P_J, U_J, \rho, \det, j)$ count those characters which do not contain U_J in their kernel.*

Proposition 1.3 *For any $\rho \in \mathrm{Irr}(Z)$, any positive integer j, and all nonnegative integers d*

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^0(P_J, U_J, \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu) = d \\ j | \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho) \tag{1a}$$

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) = - \sum_{\substack{\mu \vdash n \\ n'(\mu) = d \\ j | \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho). \tag{1b}$$

In Part I we proved Equation 1a. In the rest of this paper we prove Equation 1b and hence complete the proof of Theorem 1.1.

2 Modules for Parabolic Subgroups

The first step in the process of finding a formula for $k_d^1(P_J, U_J, \rho, \det, j)$ is to examine orbits in submodules afforded by quotient groups in the normal series for U_J discussed below. We are faced with the task of describing certain irreducible characters of parabolic subgroups P_J in $\mathrm{U}_n(q)$ which do not contain U_J in their kernels. In general this is not an easy task as $\mathrm{Irr}(P_J)$ is not known. We avoid this difficulty by examining P_J orbits of irreducible U_J characters.

In order to distinguish between parabolic subgroups in general linear and in unitary groups of varying dimensions, we make the following definition.

Definition 2.1 *1. Let P_J^{+n} denote a parabolic subgroup in $\mathrm{GL}_n(q^2)$, where $J \subseteq [n-1]$.*

2. Let P_J^n denote a parabolic subgroup in $\mathrm{U}_n(q)$, where $n = 2m$, or $2m+1$ and $J \subseteq [m]$.

Write $J = \{j_1, j_2, \ldots, j_s\}$ in increasing order.

Definition 2.2 *The following subsets of J are defined*

1. *$J(\geq j_i) = \{j_i, j_{i+1}, \ldots, j_s\}$ and $J(> j_i) = \{j_{i+1}, \ldots, j_s\}$*
2. *$J(\leq j_i) = \{j_1, j_2, \ldots, j_i\}$ and $J(< j_i) = \{j_1, j_2, \ldots, j_{i-1}\}$.*

The unipotent radical U_J has the following normal series:

$$U_J = U_{J(\geq j_1)} > U_{J(\geq j_2)} > \cdots > U_{J(\geq j_s)} = U_{j_s} \geq Z(U_{j_s}) > 1 \qquad (2)$$

The quotient groups in this series are abelian. The set of all irreducible characters of P_J divides into subsets of characters in the following way. Let χ be an irreducible character of P_J then there exits a group N in the above series such that the kernel of χ contains N but does not contain the previous subgroups in the series. Since $N \trianglelefteq P_J$ and $N \subseteq \ker(\chi)$ for $\chi \in \mathrm{Irr}(P_J)$, we may consider χ as an irreducible character of the quotient group P_J/N. There are four types of characters originally found by KU which we now briefly describe. What distinguishes these four types is the action induced by conjugation by P_J on the abelian quotients in (2).

1. *Levi Characters:* Suppose $\chi \in \mathrm{Irr}(P_J)$ and U_J is contained in $\ker(\chi)$. Then we may consider χ a character of the quotient group P_J/U_J which is of course isomorphic to L_J. Such a character χ is trivial on U_J and has already been accounted for in 1a.

2. *General Linear Characters:* Suppose $\chi \in \mathrm{Irr}(P_J)$ and for fixed i satisfying $1 \leq i < s$ we have $U_{J(\geq j_{i+1})} \subseteq \ker(\chi)$ but $U_{J(\geq j_i)} \nsubseteq \ker(\chi)$. We may consider χ as a character of $P_J/U_{J(\geq j_{i+1})}$. Then

$$P_J/U_{J(\geq j_{i+1})} \cong P^{+j_{i+1}}_{J(\leq j_i)} \times L_{J'} \quad ([15], 7.1.2.2)$$

where $J' = \{j - j_{i+1} | j \in J(> j_{i+1})\}$ and $L_{J'}$ is a levi subgroup in $\mathrm{U}_{n-2j_{i+1}}(q)$. Let $V(j_i, j_{i+1})$ denote the quotient group $U_{J(\geq j_i)}/U_{J(\geq j_{i+1})}$.

$$P^{+j_{i+1}}_{J(\leq j_i)} \cong \left(P^{+j_i}_{J(<j_i)} \times \mathrm{GL}_{j_{i+1}-j_i}(q)\right) \ltimes V(j_i, j_{i+1}).$$

We call $V(j_i, j_{i+1})$ a general linear module to indicate that a general linear group is acting on the module. The character χ restricted to $V(j_i, j_{i+1})$ is nontrivial. Hence $\chi = \chi_1\chi_2$ where χ_1 corresponds to some irreducible character of $V(j_i, j_{i+1})$ and χ_2 is an irreducible character of the factor $L_{J'}$.

3. *Unitary Linear Characters:* Now we suppose that $\chi \in \mathrm{Irr}(P_J)$ with $Z(U_{j_s}) \subseteq \ker(\chi)$ but $U_{j_s} \nsubseteq \ker(\chi)$. We may consider χ as a character of

$$P_J/Z(U_{j_s}) \cong \left(P^{+j_s}_{J(<j_s)} \times \mathrm{U}_{n-2j_s}(q)\right) \ltimes (U_{j_s}/Z(U_{j_s})) \quad ([15], 7.1.2.3).$$

Then χ corresponds to an irreducible character of the quotient $U_{j_s}/Z(U_{j_s})$, a unitary linear module to indicate that a unitary group is acting on the module.

4. *Central Characters:* Finally we make take $\chi \in \mathrm{Irr}(P_J)$ with only the trivial subgroup contained in $\ker(\chi)$ so that $Z(U_{j_s}) \nsubseteq \ker(\chi)$. In this case χ corresponds to a non trivial character of $Z(U_{j_s})$.

As mentioned these four types are outlined by Ku in [15]. The main difference in this paper is the introduction of two extra parameters in the manner of Sukizaki's approach [22] to the special linear case. Amazingly, it turns out that parabolic characters as parameterized by Ku are very well behaved with regard to their splitting upon restriction to the kernel of the determinant map. This is very convenient and not an obvious fact. One important result of this fact is that Ku's parametrization of the character q-heights is sufficient.

Conjugation by P_J on the abelian quotients in (2) gives rise to the aforementioned internal modules with non-trivial action by a group H_J in a quotient group P_J/N of P_J, where N appears in the series (2). Hence we need to examine what occurs at the level of H_J.

2.1 Parabolic Actions

In this section we study the modules which arise in the following situation. Fix positive integers n_1 and n_2. Let V_1 be the natural module for $\mathrm{GL}_{n_1}(q^2)$ and V_2 the dual of the natural module for $\mathrm{GL}_{n_2}(q^2)$. Fix a basis for V_1, n_1-dimensional column vectors $\{e_1, e_2, \ldots, e_{n_1}\}$ where e_i has a 1 in the i-position and zeros elsewhere. Fix a basis for V_2, n_2-dimensional row vectors $\{e^1, e^2, \ldots, e^{n_2}\}$ where e^j has a 1 in the j-position and zeros elsewhere. Set $V = V_1 \otimes V_2 \cong M_{n_1,n_2}(q^2)$. A basis for V is given by $\{E_{i,j}\}$ where $E_{i,j} = e_i \otimes e^j$ the $n_1 \times n_2$-matrix with (i,j)-entry 1 and zeros elsewhere. Then $\mathrm{GL}_{n_1}(q^2) \times \mathrm{GL}_{n_2}(q^2)$ acts on V in the natural way via left multiplication by $\mathrm{GL}_{n_1}(q^2)$

and right multiplication by inverses in $\mathrm{GL}_{n_2}(q^2)$. Let G_i be a subgroup of $\mathrm{GL}_{n_i}(q^2)$. Then $G = G_1 \times G_2$ acts on V and hence V is a module for G. In subsequent sections we will be considering the following cases:

1. The group $G_1 = P_J^{+n_1}$ and $G_2 = \mathrm{GL}_{n_2}(q^2)$,
2. the group $G_1 = P_J^{+n_1}$ and $G_2 = \mathrm{U}_{n_2}(q)$, and
3. the group $G_1 = P_J^{+n_1}$ and G_2 is an isomorphic copy of G_1. In this case, as we will see, the module we consider is a subgroup of V isomorphic to $M_{n_1,n_2}(q)$. We will discuss this in the central module section.

The vector space V is an elementary abelian group. We have the following G-isomorphisms of abelian groups ([15], 6.1.2):

$$\mathrm{Irr}(V) = \hom(V, \mathbb{C}^*) \cong \hom(V, \mathbb{C}_p) \cong \hom_{F_p}(V, F_p) \cong \hom_{F_{q^2}}(V_1 \otimes V_2, F_{q^2}) \cong \hom_{F_{q^2}}(V_1, V_2^*)$$

where now V_2^* is the restriction to G_2 of the natural module for $\mathrm{GL}_{n_2}(q^2)$.

The first isomorphism is clear since complex characters of V take values in $\mathbb{C}_p$. The second isomorphism is also clear since the multiplicative group $\mathbb{C}_p$ can be identified with the additive group F_p. The last isomorphism is also clear by adjoint associativity of the tensor product. The penultimate isomorphism is less clear. The field F_{q^2} is a finite extension of F_p. Let θ generate F_{q^2} so that $\{1, \theta, \theta^2, \ldots\}$ is a basis for F_{q^2} as a vector space over F_p. Note that $\theta^0 = 1$. Define the projection

$$\pi : F_{q^2} \longrightarrow F_p \ , \qquad \sum_i a_i \theta^i \mapsto a_0.$$

We fixed a basis $\{E_{i,j}\}$ for V over F_{q^2} above. Let U be the F_p-span of $\{E_{i,j}\}$. Then

$$V = F_{q^2} \otimes_{F_p} U$$

so that $\{\theta^k \otimes E_{i,j}\}$ is an F_p-basis of V. The following is a well-defined isomorphism

$$\hom_{F_{q^2}}(V, F_{q^2}) \longrightarrow \hom_{F_p}(V, F_p) \ , \qquad f \mapsto \pi \circ f.$$

Hence we have $\hom_{F_p}(V, F_p) \cong \hom_{F_{q^2}}(V_1 \otimes V_2, F_{q^2})$.

The action of G on V gives rise to a parallel action of G on $\mathrm{Irr}(V)$ and hence on $\hom_{F_{q^2}}(V_1, V_2^*)$. Since we fixed a basis for V_2, the dual V_2^* has basis $\{(e^1)^*, (e^2)^*, \ldots, (e^{n_2})^*\}$ where $(e^j)^*$ is the n_2-dimension row vector with 1 in the j-position and zeros elsewhere. Take $\tau \in \mathrm{Irr}(V)$. Then τ corresponds to $f \in \hom_{F_{q^2}}(V_1, V_2^*)$ and

$$V_1 / \ker(f) \cong f(V_1) \cong (V_2/\mathrm{Ann}(f))^* .$$

where

$$\ker(f) = \{v \in V_1 \mid f(v)(w) = 0 \ \forall w \in V_2\} \text{ is the kernel of } f, \text{ and}$$
$$\mathrm{Ann}(f) = \{w \in V_2 \mid f(v)(w) = 0 \ \forall v \in V_1\} \text{ is the annihilator of } f(V_1) \text{ in } V_2.$$

The codimension of both of these subspaces is the same. If r is the codimension of $\ker(f)$ and $\mathrm{Ann}(f)$ we will say that f has rank r. In this way if $\tau \in \mathrm{Irr}(V)$ corresponds to $f \in \hom_{F_{q^2}}(V_1, V_2^*)$ then τ is said to have rank r. We remark that via the isomorphism between $\mathrm{Irr}(V)$ and V, if τ corresponds to the matrix v then the rank of τ is the row rank of v, which is invariant under the action of G.

We are interested in describing representatives of G-orbits in $\mathrm{Irr}(V)$ and their stabilizers in G. Given $\tau \in \mathrm{Irr}(V)$, let $\ker(\tau)$ and $\mathrm{Ann}(\tau)$ denote $\ker(f)$ and $\mathrm{Ann}(f)$, respectively, where τ corresponds to the element $f \in \hom_{F_{q^2}}(V_1, V_2^*)$. Write $\overline{V}_1 = V_1/\ker(\tau)$ and $\overline{V}_2 = V_2/\mathrm{Ann}(\tau)$. The dimension of both quotient spaces is r. Let

$$C_{G_i}(\overline{V}_i) = \{\ g \in G_i \mid g \cdot \overline{v} = \overline{v} \ \forall \overline{v} \in \overline{V}_i\ \}$$

where as noted above $g \cdot \overline{v}$ indicates left or right multiplication depending on i equal to 1 or 2, respectively. As stated by Ku ([15], p.67) we have

$$C_{G_1}(\overline{V}_1) \times C_{G_2}(\overline{V}_2) \leq T_G(\tau) \leq T_{G_1}(\ker(\tau)) \times T_{G_2}(\mathrm{Ann}(\tau)).$$

Before describing these orbits and stabilizers we pause to consider some cancellation which occurs in the alternating sum of DOC.

2.2 Some Useful Cancellation

An alternating sum may reduce to a smaller alternating sum in an advantageous way. One may approach this from a combinatorial perspective in which case, like terms that appear with opposite parity cancel one another. We will make use of this approach later in the paper. One may also approach the reduction of an alternating sum from a topological perspective. We discuss this now and apply the results in order to make a first reduction of 1b.

Consider the Burnside ring $b(G)$ of a finite group G the free abelian group on equivalence classes $[G/H]$, where $[G/H]$ is equal to $[G/K]$ if and only if H and K are conjugate subgroups of G. A typical element of $b(G)$ is of the form

$$a_1[G/H_1] + a_2[G/H_2] + \cdots + a_N[G/H_N]$$

where $a_i \in \mathbb{Z}$ and the H_i are representatives of conjugacy classes of subgroups in G. Multiplication in $b(G)$ is given by

$$[G/H] \cdot [G/K] = [G/(H \cap K)].$$

Let G act on a finite poset P ordered by inclusion. The simplicial complex $\mathcal{O}(P)$ consists of chains,

$$c : x_0 < x_1 < \cdots < x_k \qquad x_i \in P.$$

where we require strict inclusion. The chain c as above has length $k+1$. The chains of length $k+1$ form the k-simplices of $\mathcal{O}(P)$. By convention the -1-simplex is 1 the trivial G-set. Let $\Delta(P) = \{1 < x_0 < x_1 < \cdots < x_k \mid x_i \in P\}$. Every chain in $\Delta(P)$ in (including the trivial chain) begins with the trivial G-set 1. For $c : 1 < x_0 < x_1 < \cdots < x_k \in \Delta(P)$ define the absolute value $|c| = k+1$. The reduced Lefshetz element of P in G is an element of $b(G)$ and is defined

$$\Lambda_G(P) = \sum_{c \in \Delta(P)/G} (-1)^{|c|}[G/G_c]$$

where c runs over a set of G-orbit representatives in $\Delta(P)$ and G_c is the stabilizer $T_G(c)$ of c in G.

We may assign topological concepts to posets in the following sense. When we say a poset P has a certain property we mean that its associated simplicial complex has the property. See ([18]) for a discussion of this.

Let G act on posets P and Q. A poset map $f : P \longrightarrow Q$ is a map that preserves ordering, i.e. $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$. The poset map f gives rise to a simplicial map $f_{\mathcal{O}} : \mathcal{O}(P) \longrightarrow \mathcal{O}(Q)$. This map is G-equivariant if $gf_{\mathcal{O}}(c) = f_{\mathcal{O}}(gc)$ for all $c \in \mathcal{O}(P)$. Two simplicial maps $f_{\mathcal{O}}, g_{\mathcal{O}} : \mathcal{O}(P) \longrightarrow \mathcal{O}(Q)$ are homotopic if there exists a continuous map $H : \mathcal{O}(P) \times [0,1] \longrightarrow \mathcal{O}(Q)$ such that $H(c,0) = f_{\mathcal{O}}(c)$ and $H(c,1) = g_{\mathcal{O}}(c)$ for all $c \in \mathcal{O}(P)$. If f, $g : P \longrightarrow Q$ are poset maps such that $f(x) \leq g(x)$, for all $x \in P$, then $f_{\mathcal{O}}$ and $g_{\mathcal{O}}$ are homotopic.

The spaces $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are said to be G-homotopy equivalent if there exist G-equivariant maps $f_{\mathcal{O}} : \mathcal{O}(P) \to \mathcal{O}(Q)$ and $g_{\mathcal{O}} : \mathcal{O}(Q) \to \mathcal{O}(P)$ such that $f_{\mathcal{O}} \circ g_{\mathcal{O}}$, $id_{\mathcal{O}(P)}$ are G-equivariant homotopic and $g_{\mathcal{O}} \circ f_{\mathcal{O}}$, $id_{\mathcal{O}(Q)}$ are G-equivariant homotopic ([25], p.352). We may say that the posets P and Q are G-homotopy equivalent, by which we mean that $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are G-homotopy equivalent. Given our interest in the reduced Lefshetz element we may also say that that $\Delta(P)$ and $\Delta(Q)$ are G-homotopy equivalent by which we mean again that $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are G-homotopy equivalent.

It is well known that, if $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are G-homotopy equivalent, then $\Lambda_G(P) = \Lambda_G(Q)$.

Let $S(G)$ denote the set of subgroups of G. Let $f : S(G) \longrightarrow \mathbb{Z}$ be a G-stable map, i.e. constant on conjugate subgroups of G. Set $f([G/G_c]) = f(G_c)$ and extend linearly to $b(G)$. We may apply f to $\Lambda_G(P)$,

$$f(\Lambda_G(P)) = \sum_{c \in \Delta(P)/G} (-1)^{|c|} f(G_c).$$

This is an integer. Moreover if P and Q are G-homotopy equivalent then $f(\Lambda_G(P)) = f(\Lambda_G(Q))$, i.e.

$$\sum_{c \in \Delta(P)/G} (-1)^{|c|} f(G_c) = \sum_{c \in \Delta(Q)/G} (-1)^{|c|} f(G_c).$$

Given a G-poset P, we may make use of a contractible sub-poset in order to find a smaller complex which is G-homotopy equivalent to $\mathcal{O}(P)$. A poset Q is G-contractible if Q is itself G-homotopy equivalent to a point. Let Q contain a single element x. Then $\Delta(Q)$ consists of two chains

$$c_1 : 1 \text{ with } |c_1| = 0 \text{ and } c_2 : 1 < x \text{ with } |c_2| = 1,$$

which are both necessarily stabilized by G. Observe that $\Lambda_G(Q) = [G/G] - [G/G] = 0$ and thus for any G-stable map f, $f(\Lambda_G(Q)) = 0$

A reference for more material on G-homotopy is ([21], Chapters 3 and 4).

An important example of homotopy equivalence: Let V be a vector space over a finite field, and let P be the poset of non-trivial proper subspaces of V ordered by inclusion. Let w be a non-trivial proper subspace of V. Let Q be the sub-poset of P whose elements are subspaces which are not complements in V to w. Then $\Delta(P)$ is the set of chains of subspaces of V beginning with the zero subspace and $\Delta(Q)$ is the subset of chains which do not contain a complement in V to w. Let $G = T_{\mathrm{GL}(V)}(w)$. The group G certainly acts on P and Q. The vector space w is in Q and is fixed by G. It turns out that Q is G-contractible ([24], Corollary 1.9). Thus $\Lambda_G(Q) = 0$. Let $\Delta(P, w)$ denote the sub-set of $\Delta(P)$ consisting of chains which contain complements to w. Then $\Delta(P) = \Delta(Q) \bigsqcup \Delta(P, w)$ ([15], Lemma 5.2.13). Since Q is G-contractible, $\Lambda_G(P) = \Lambda_G(P, w)$.

We are now ready to apply this discussion to our situation. For the rest of this section let $G = \mathrm{GL}_{n_1}(q^2) \times G_2$ where G_2 is a subgroup of $\mathrm{GL}_{n_2}(q^2)$. For now, we will be considering G_2 to be either $\mathrm{GL}_{n_2}(q^2)$ or $\mathrm{U}_{n_2}(q)$. Let $H = G \ltimes V$ where $V = V_1 \otimes V_2$ where V_1 is the natural module for $\mathrm{GL}_{n_1}(q^2)$ and V_2 is the dual of the natural module for $\mathrm{GL}_{n_1}(q^2)$, as above. For each $J \subseteq [n_1 - 1]$, we let G_J denote $P_J^{+n_1} \times G_2$ and H_J denote $G_J \ltimes V$.

Definition 2.3 *Let* $\mathrm{Irr}(V, r)$ *denote the subset of characters in* $\mathrm{Irr}(V)$ *of rank* r*, where we recall that the rank of* τ *is the codimension of* $\ker(\tau)$ *in* V_1*, viewing* τ *as a homomorphism from* V_1 *to* V_2^**.*

Set $X = \mathrm{Irr}(V, r)$. Let $Y = \{y \leq V_1 \mid \mathrm{codim}(y) = r\}$ and for $y \in Y$ let $X(y) = \{\tau \in X \mid \ker(\tau) = y\}$. We extend the action of $\mathrm{GL}_{n_1}(q^2)$ on V_1 to H by declaring that $G_2 \rtimes V$ act trivially on V_1. In this way H acts on X and Y but also on the poset $P = P(V_1)$ of subspaces of V_1 ordered by inclusion. Then $\Delta(P)$ is the set of flags in P together with the empty flag and so H also acts on $\Delta(P) \times X$. A set of orbit representatives for the action of $\mathrm{GL}(V_1)$ and hence H on $\Delta(P)$ is given by $\{c_J \mid J \subseteq [n_1-1]\}$, where if $J = \{j_1, j_2, \ldots, j_s\}$ then c_J is the flag

$$0 < V_{j_1} < V_{j_2} < \cdots < V_{j_s}$$

where V_{j_i} is the subspace of V_1 spanned by the vectors $\{e_1, e_2, \ldots, e_{j_i}\}$. We also note that $P_J^{+n_1}$ is the stabilizer of the flag c_J.

The proposition below is a modification of ([15], Proposition 6.2.1), where we obtain a first reduction for equation 1b. Ku's proof makes use of some fairly elaborate abstractions involving alternating sums. We present a more direct proof with the necessary adjustments which include extra parameters in the definition of an H-stable function which lead to the desired reduction. The significance of this result is that for fixed r we need only describe orbit representatives for the subset $X(w) = \{\tau \mid \ker \tau = w\}$ in $\mathrm{Irr}(V, r)$, where w is a complement in V_1 of the r-dimensional subspace stabilized by $P_r^{+n_1}$. In fact, since $T_{H_J}(\tau) \leq T_{H_J}(w) \leq H_J$ and $T_{H_J}(w) = T_{G_J}(w) \ltimes V$ by Corollary 2.14 from [1] it is sufficient to find representatives for $T_{G_J}(w)$-orbits in $X(w)$. What is more we only need the stabilizers in H_J where $J \cup \{n_1\}$ contains the element r.

We will be considering the group $H_J = G_J \ltimes V$ as embedded in $\mathrm{U}_n(q)$.

Definition 2.4 *Define the map on G_J,*

$$\mathcal{D} : P_J^{+n_1} \times G_2 \longrightarrow F_{q^2}^* \ , \qquad (A, B) \mapsto \det(A)^{i(1-q)} \det(B)^k,$$

for some positive integer i where the exponent of $\det(B)$ depends on G_2 in the following way:

$$k = \begin{cases} 1-q, & \text{if } G_2 = \mathrm{GL}_{n_2}(q^2); \\ 1, & \text{if } G_2 = \mathrm{U}_{n_2}(q). \end{cases}$$

Extend $\mathcal{D}$ to H_J by letting $\mathcal{D}(v) = 1$ for all $v \in V$. Observe that $\ker(\mathcal{D})$ is normal in H_J, contains V, and the quotient $H_J/\ker(\mathcal{D})$ is cyclic. The map $\mathcal{D}$ is constructed to be the restriction to H_J of the determinant map on $\mathrm{U}_n(q)$. The value of the integer i depends on the embedding of H_J in $\mathrm{U}_n(q)$.

Definition 2.5 *For a subset $X \subseteq \mathrm{Irr}(V)$, let $k_d(H_J, X, \rho, \mathcal{D}, j)$ denote the number of irreducible characters χ of H_J of q-height d, lying over ρ, and corresponding to $\tau \in X$ such that χ restricted to the kernel of the map $\mathcal{D}$ is a sum of j' irreducible characters where j divides j'.*

Proposition 2.6 *Let $Z \leq Z(G)$ and $\rho \in Irr(Z)$. Fix $1 \leq r \leq \min(n_1, n_2)$. Let $X = \mathrm{Irr}(V, r)$ and Let $Y = \{y \leq V_1 \mid \mathrm{codim}(y) = r\}$. Let w be a complement in V_1 to the r-dimensional subspace stabilized by $P_r^{+n_1}$, in its action on the natural module for $\mathrm{GL}_{n_1}(q^2)$. Let $X(w) = \{\tau \in X \mid \ker(\tau) = w\}$. Let $\mathcal{D}$ be the map on H_J defined above. For $d \geq 0$ the following holds*

1. *If* $r = n_1$*, then* $Y = \{0\}$ *and* $X = X(0)$*.*
2. *If* $r < n_1$*, then*
$$\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho, \mathcal{D}, j) = \sum_{\substack{J\subseteq[n_1-1]\\ r\in J}} (-1)^{|J|} k_d(H_J, X(w), \rho, \mathcal{D}, j).$$

Proof: The first part is immediate since the trivial subspace is the only subspace of V_1 of codimension n_1. The proof of the second part is based on the exposition on homotopy equivalences which begins this section.

Let $r < n_1$. We define a function

$$f : \Delta(P) \times Y \longrightarrow \mathbb{Z}\ , \quad (c_J, y) \mapsto k_d(H_J, X(y), \rho, \mathcal{D}, j).$$

This is a well defined H-stable function on $\Delta(P) \times Y$, i.e. constant on the H-orbits. Indeed, the action of H on $\Delta(P)$ preserves chain type and thus a set of H-orbit representatives is given by

$$\{\ c_J \mid \in J \subseteq [n_1 - 1]\}.$$

Moreover, $k_d(H_J, \tau, \rho, \mathcal{D}, j) = k_d(H_J, {}^h\tau, \rho, \mathcal{D}, j)$ for any $h \in H_J$. If $y' = gy$ for $g \in P_J^{+n_1}$, then

$$\begin{aligned}
k_d(H_J, X(y'), \rho, \mathcal{D}, j) &= \sum_{\ker\tau=y'} k_d(H_J, \tau, \rho, \mathcal{D}, j)\\
&= \sum_{\ker\tau=gy} k_d(H_J, \tau, \rho, \mathcal{D}, j)\\
&= \sum_{g^{-1}\ker\tau=y} k_d(H_J, {}^{g^{-1}}\tau, \rho, \mathcal{D}, j)\\
&= \sum_{\ker\tau=y} k_d(H_J, \tau, \rho, \mathcal{D}, j)\\
&= k_d(H_J, X(y), \rho, \mathcal{D}, j).
\end{aligned}$$

The function f induces the following projection functions. For fixed $c_J \in \Delta(P)$

$$f_{c_J} : Y \longrightarrow \mathbb{Z}\ , \quad y \mapsto k_d(H_J, X(y), \rho, \mathcal{D}, j)$$

is $T_H(c_J) = H_J$-stable on Y; for fixed $y \in Y$

$$f_y : \Delta(P) \longrightarrow \mathbb{Z}\ , \quad c_J \mapsto k_d(H_J, X(y), \rho, \mathcal{D}, j)$$

is $T_H(y)$-stable on $\Delta(P)$.

For fixed $\tau \in X(y)$, $T_{H_J}(\tau) \le T_{H_J}(y) \le H_J$. Thus by Corollary 2.14 in [1]

$$\begin{aligned}
k_d(H_J, X(y), \rho, \mathcal{D}, j) &= \sum_{\tau\in X(y)/H_J} k_d(H_J, \tau, \rho, \mathcal{D}, j)\\
&= \sum_{\tau\in X(y)/T_{H_J}(y)} k_{d-d'}(T_{H_J}(y), \tau, \rho, \mathcal{D}, j')\\
&= k_{d-d'}(T_{H_J}(y), X(y), \rho, \mathcal{D}, j')
\end{aligned}$$

where d' is the exponent of q in $|T_{H_J}(y)\backslash H_J|$ and j' is the least positive integer such that

$$j|j' \cdot |T_{H_J}(y)\ker(\mathcal{D})\backslash H_J|.$$

In all three cases which we will consider $j' = j$ will hold. Indeed, if $G_2 = \mathrm{GL}_{n_2}(q^2)$, or $\mathrm{U}_{n_2}(q)$ then this is certainly true since $n_2 \neq 0$ so

$$T_{H_J}(y) = \left(T_{P_J^{+n_1}} \times G_2\right) \ltimes V,$$

hence $\mathcal{D}(T_{H_J}(y)) = \mathcal{D}(H_J)$. We will be considering the central case where G_2 is an isomorphic copy of $P_J^{+n_1}$. In this instance, the fact that $r < n_1$ is sufficient to imply that $j' = j$.

Let d_0 be the exponent of q in $|H|$. Let w be a subspace of V_1 as defined in the statement of the proposition. Define the following function on subgroups of $T_H(w)$

$$g : S(T_H(w)) \longrightarrow \mathbb{Z}$$

$$K \mapsto \begin{cases} k_{d-(d_0-d(K))}(K, X(w), \rho, \mathcal{D}, j), & \text{if } V \leq K; \\ 0, & \text{otherwise.} \end{cases}$$

where $d(K)$ is the exponent of q in $|K|$. This is a $T_H(w)$-stable function. For $c_J \in \Delta(P)$, $T_H(c_J) = H_J$. Since the q-height of H is equal to the q-height of H_J,

$$d_0 - d(K) = |T_{H_J}(w)\backslash H_J|.$$

Hence $g(H_J) = f_w(c_J)$. Applying g to the reduced Lefshetz element we have

$$g(\Lambda_{T_H(w)}(P)) = g(\Lambda_{T_H(w)}(P, w)).$$

We put the above together

$$\begin{aligned}
&\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho, \mathcal{D}, j) \\
&\quad = \sum_{c_J\in\Delta(P)/H} (-1)^{|J|} \sum_{y\in Y/H_J} k_d(H_J, X(y), \rho, \mathcal{D}, j) \\
&\quad = \sum_{y\in Y/H} \sum_{c_J\in\Delta(P)/T_H(y)} (-1)^{|J|} k_d(H_J, X(y), \rho, \mathcal{D}, j) \\
&\quad = \sum_{c_J\in\Delta(P)/T_H(y)} (-1)^{|J|} k_d(H_J, X(w), \rho, \mathcal{D}, j), \text{ since there is a unique } H\text{-orbit in } Y \\
&\quad = \sum_{c_J\in\Delta(P)/T_H(w)} (-1)^{|J|} k_{d-d'}(T_{H_J}(w), X(w), \rho, \mathcal{D}, j) \\
&\quad = \sum_{c_J\in\Delta(P,w)/T_H(w)} (-1)^{|J|} k_{d-d'}(T_{H_J}(w), X(w), \rho, \mathcal{D}, j) \\
&\quad = \sum_{c_J\in\Delta(P,w)/T_H(w)} (-1)^{|J|} k_d(H_J, X(w), \rho, \mathcal{D}, j) \\
&\quad = \sum_{\substack{J\subseteq[n_1-1] \\ r\in J}} (-1)^{|J|} k_d(H_J, X(w), \rho, \mathcal{D}, j),
\end{aligned}$$

and we are done.

2.3 General Linear Modules

In this section we first summarize Ku's results and then examine a map which is related to the usual determinant map. In this section G_1 is $P_J^{+n_1}$, a parabolic subgroup of $\mathrm{GL}_{n_1}(q^2)$ for fixed $J \subset [n_1-1]$. Let G_2 be $\mathrm{GL}_{n_2}(q^2)$. The module V_1 is the restriction to $P_J^{+n_1}$ of the natural module for $\mathrm{GL}_{n_1}(q^2)$ and V_2 is the dual of the natural module for $\mathrm{GL}_{n_2}(q^2)$. Recall, in this context $G_J = P_J^{+n_1} \times \mathrm{GL}_{n_2}(q^2)$. Fix $r \in J$ or $r = n_1$. If $r < n_1$ then let $w \leq V_1$ be a complement to the r-dimensional subspace stabilized by $P_r^{+n_1}$. If $r = n_1$ then let $w \leq V_1$ be the trivial subspace. Let $\tau \in X(w)$ so that τ has rank r and $\ker \tau = w$. We can use r to divide the set J into two subsets:

$$\{j \in J \mid j < r\} \quad \text{and} \quad \{j \in J \mid r < j\}.$$

Let us set $J_2 = \{j \mid j \in J \text{ and } j < r\}$ and $J_1 = \{j - r \mid j \in J \text{ and } r < j\}$. Notice that we have the containment $J_2 \subseteq [r-1]$ and $J_1 \subseteq [n_1 - r]$.

Proposition 2.7 *([15], Lemma 6.2.2) The group $T_{G_J}(w) = P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r} \times \mathrm{GL}_{n_2}(q^2)$ and is transitive on $X(w)$. The stabilizer of τ in G_J is*

$$T_{G_J}(\tau) = \begin{cases} P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+n_2}, & \text{if } r = n_2; \\ P_{J_1}^{+(n_1-r)} \times P_{J_2 \cup \{r\}}^{+n_2}, & \text{if } r < n_2. \end{cases} \tag{3}$$

2.4 The determinant map in the general linear module context

We would like to determine the determinant map on

$$G_J = P_J^{+n_1} \times \mathrm{GL}_{n_2}(q^2),$$

as embedded in a larger unitary group. At this stage we consider the map

$$\begin{aligned} \mathcal{D} : P_J^{+n_1} \times \mathrm{GL}_{n_2}(q^2) &\longrightarrow \quad F_{q^2}^* \\ (A, B) \quad &\mapsto (\det(A)^i \det(B))^{1-q}, \end{aligned}$$

for some positive integer i. As we will see $\mathcal{D}$ is constructed to be the restriction to G_J of the usual determinant map on $\mathrm{U}_n(q)$. The integer i depends on the embedding of G_J in $\mathrm{U}_n(q)$. For the rest of this subsection let $K = \ker(\mathcal{D})$

Our first observation is that the image of G_J under this map is all of $\mathbb{C}_{(q+1)}$ since $n_2 \neq 0$. We need to examine the map $\mathcal{D}$ restricted to the subgroup $T_{G_J}(\tau)$ and calculate $|T_{G_J}(\tau)K \backslash G_J|$. The cases depend on r, n_1, and n_2. In order to calculate $|T_{G_J}(\tau)K \backslash G_J|$ we need to find the image of $T_{G_J}(\tau)$ under $\mathcal{D}$. If $\mathcal{D}(T_{G_J}(\tau)) = \mathbb{C}_{(q+1)/h}$ then $\mathcal{D}(T_{G_J}(\tau)K) = \mathbb{C}_{(q+1)/h}$ so that

$$|T_{G_J}(\tau)K \backslash G_J| = |\mathbb{C}_{(q+1)/h}| \backslash |\mathbb{C}_{(q+1)}| = h.$$

We examine the four cases:

1. If $r = n_1 = n_2$ then $T_{G_J}(\tau) = P_J^{+r}$ with
$$\begin{array}{rccc} \mathcal{D}: & P_J^{+r} & \longrightarrow & F_{q^2}^* \\ & B & \mapsto & (\det(B)^{i+1})^{1-q}. \end{array}$$
Also notice that $\mathcal{D}(P_J^{+r}) = \mathbb{C}_{(q+1)/\gcd(q+1,i+1)}$.

2. If $r = n_1$ but $r < n_2$ then $T_{G_J}(\tau) = P_{J\cup\{r\}}^{+n_2} = \left(P_J^{+r} \times \mathrm{GL}_{n_2-r}(q^2)\right) \ltimes M_{r,n_2-r}(q^2)$. Then $\mathcal{D}$ is trivial on the normal factor $M_{r,n_2-r}(q^2)$ and
$$\begin{array}{rccc} \mathcal{D}: & \left(P_J^{+r} \times \mathrm{GL}_{n_2-r}(q^2)\right) & \longrightarrow & F_{q^2}^* \\ & (B,C) & \mapsto & (\det(B)^{i+1}\det(C))^{1-q}. \end{array}$$
Observe that $\mathcal{D}(\left(P_J^{+r} \times \mathrm{GL}_{n_2-r}(q^2)\right) \ltimes M_{r,n_2-r}(q^2)) = \mathbb{C}_{(q+1)}$ since $n_2 - r \neq 0$.

3. If $r < n_1$ but $r = n_2$ then $T_{G_J}(\tau) = P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r}$ with
$$\begin{array}{rccc} \mathcal{D}: & P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r} & \longrightarrow & F_{q^2}^* \\ & (A,B) & \mapsto & (\det(A)^i(\det(B)^{i+1})^{1-q}. \end{array}$$
Take any element α of $F_{q^2}^*$. Then the diagonal matrix $A = (\alpha^{-1}, 1, 1, \cdots, 1)$ is in $P_{J_1}^{+(n_1-r)}$ and the diagonal matrix $B = (\alpha, 1, 1, \cdots, 1)$ is in $P_{J_2}^{+r}$. Then
$$\mathcal{D}(A,B) = (\alpha^{-i}\alpha^{i+1})^{1-q} = \alpha^{1-q}$$
and thus $\mathcal{D}(P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r})$ is all of $\mathbb{C}_{(q+1)}$.

4. Finally we suppose that $r < n_1$ and $r < n_2$. We have
$$\begin{aligned} T_{G_J}(\tau) &= P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+n_2} \\ &= P_{J_1}^{+(n_1-r)} \times \left(P_{J_2}^{+r} \times \mathrm{GL}_{n_2-r}(q^2)\right) \ltimes M_{r,n_2-r}(q^2). \end{aligned}$$
As above, $\mathcal{D}$ is trivial on the normal factor $M_{r,n_2-r}(q^2)$ and
$$\begin{array}{rccc} \mathcal{D}: & P_{J_1}^{+(n_1-r)} \times P_J^{+r} \times \mathrm{GL}_{n_2-r}(q^2) & \longrightarrow & F_{q^2}^* \\ & (A,B,C) & \mapsto & (\det(A)^i\det(B)^{i+1}\det(C))^{1-q}. \end{array}$$
In this case we also have $\mathcal{D}(T_{G_J}(\tau))$ equal to all of $\mathbb{C}_{(q+1)}$ since $n_2 - r \neq 0$.

Summarizing these results, we get the following proposition which is a step in computing the number of irreducible characters of P_J (in an appropriate p-block, of appropriate q-height) which split as desired upon restriction to $P_J \cap \mathrm{SU}_n(q)$:

Proposition 2.8
$$|T_{G_J}(\tau)K\backslash G_J| = \begin{cases} \gcd(q+1, i+1), & \textit{if } r = n_1 = n_2; \\ 1, & \textit{otherwise.} \end{cases}$$

2.5 Unitary Modules

In this section we first summarize Ku's results and then examine a map which is related to the usual determinant map. We make use of the following notations which are due to Ku: $S^u(V,J,r)$, $S^{su}(V,J,r)$, and $S^{nu}(V,J,r)$. In this section G_1 is $P_J^{+n_1}$ a parabolic subgroup of $\mathrm{GL}_{n_1}(q^2)$ for fixed $J \subset [n_1 - 1]$ as above. In the previous section we had $G_2 = \mathrm{GL}_{n_2}(q^2)$ which is transitive on subspaces of the same dimension in the dual space V_2^*. Now we consider $G_2 = \mathrm{U}_{n_2}(q)$. Recall, in this context $G_J = P_J^{+n_1} \times \mathrm{U}_{n_2}(q)$. The module V_1 is still the restriction to $P_J^{+n_1}$ of the natural module for $\mathrm{GL}_{n_1}(F_{q^2})$. However, now V_2 is the dual of the natural module for $\mathrm{U}_{n_2}(q)$ and thus has a unitary structure. In this case, G_2 is not transitive on subspaces of the same dimension of V_2^*.

Recall if U is a unitary vector space, a subspace W is totally isotropic if $< v, w >= 0$ for all vectors v, w in W, where $<,>$ indicates the hermitian form on U. A totally isotropic subspace W is degenerate since its radical $rad(W) = W \cap W^{\perp} = W$. A chain of totally isotropic subspaces will be called a singular chain. A chain of subspaces which are not all totally isotropic will be called a non-singular chain. A subspace W is non-degenerate if $rad(W) = 0$ in which case $V = W \oplus W^{\perp}$.

Let W be a non-degenerate subspace of V_2^* of dimension r. Then

$$T_{\mathrm{U}_{n_2}(q)}(W) = \mathrm{U}_{n_2-r}(q) \times \mathrm{U}_r(q).$$

Now consider W a totally isotropic subspace. A basis for V_2^* has already been fixed. To simplify notation, denote $(e^j)^*$ by e_j so that the basis is $\{e_1, e_2, \ldots, e_{n_2}\}$. Further suppose that with respect to the inner product on V_2^* we have

$$< e_i, e_j >= \begin{cases} 1, & \text{if } i+j = n_2+1; \\ 0, & \text{otherwise.} \end{cases}$$

Let W be the totally isotropic subspace equal to $< e_1, e_2, \ldots, e_r >$. Note that r must be less than or equal to $[n_2/2]$. Then the stabilizer is a maximal parabolic subgroup

$$T_{\mathrm{U}_{n_2}(q)}(W) = P_r^{n_2}.$$

Just as in the previous section we fix r where either $r \in J$ or $r = n_1$. If $r = n_1$ then let $w \leq V_1$ be the trivial subspace. If $r < n_1$ then let $w \leq V_1$ be a complement to the r-dimensional subspace stabilized by $P_r^{+n_1}$. Let $\tau \in X(w)$ so that τ has rank r and $\ker \tau = w$. Ku parameterizes the $T_{G_J}(w)$ orbits on $X(w)$ with chains of subspaces in V_2^*. These are so-called normal flags of fixed type depending on J and reflect the unitary structure in V_2^* which we define now.

Definition 2.9 *A normal flag in V_2^* is a chain of subspaces*

$$c : 0 < V_1 < V_2 < \cdots < V_s$$

satisfying the following. There exists $0 = i_0 < i_1 < \cdots i_k \leq s$, $k \geq 0$, such that for all $0 \leq j \leq k$

1. *V_{i_j} is either a non-degenerate subspace in V_2^* or the zero subspace and*

2. *for any $i_j < i < i_{j+1}$ we have $V_i = V_{i_j} \oplus rad(V_i)$, where we assume $i_{k+1} = s+1$ and $V_{s+1} = V_2^*$.*

Take linear maps f, $g : V_1 \to V_2^*$ with kernel w. The stabilizer $T_{G_J}(w) = P_{J_1}^{+n_1-r} \times P_{J(<r)}^{+r} \times \mathrm{U}_{n_2}(q)$. Under its action on $X(w)$, f and g are in the same orbit if and only if $f(c_J)$ and $g(c_J)$ are isomorphic as flags in V_2^*. Observe that $f(c_J)$ and $g(c_J)$ are both flags of type $\{J(<r) \cup \{r\}\} \backslash \{n_2\}$ because we can choose a basis so that $w =< e_{r+1}, \ldots, e_{n_1} >$ and $f(w) = g(w) = 0$ the trivial subspace. Let $\mathcal{P}(V_2^*)$ be the poset of subspaces in V_2^* ordered by inclusion.

If $r = n_1$ then $X(w) = X$ and $T_{G_J}(w) = G_J$. The G_J-orbits in X are in 1-1 correspondence with the $\mathrm{U}_{n_2}(q)$-orbits on chains of type $\{J \cup \{r\}\} \backslash \{n_2\}$ in $\mathcal{P}(V_2^*)$. If τ corresponds to chain c of such type then

$$T_{G_J}(\tau) = T_{\mathrm{U}_{n_2}(q)}(c).$$

See ([15],Lemma 6.3.1).

If $r < n_1$ then w is non-trivial and

$$T_{G_J}(w) = P_{J_1}^{+n_1-r} \times P_{J(<r)}^{+r} \times \mathrm{U}_{n_2}(q)$$

where $J_1 = \{j - r \mid j \in J(> r)\}$. The group $P_{J_1}^{+n_1-r}$ acts trivially on the quotient space $\overline{V}_1 = V_1/w$ and hence acts trivially on $X(w)$ which is isomorphic to $\mathrm{Irr}(\overline{V}_1 \otimes V_2, r)$. By the above discussion for the case $r = n_1$ the $(P_{J(<r)}^{+r} \times \mathrm{U}_{n_2}(q))$-orbits in $\mathrm{Irr}(\overline{V}_1 \otimes V_2, r)$ are in 1-1 correspondence with the $\mathrm{U}_{n_2}(q)$-orbits on chains of type $\{J(<r) \cup \{r\}\} \backslash \{n_2\}$ in $\mathcal{P}(V_2^*)$.

Definition 2.10 *Let $S^u(V, J, r)$ denote the set of $T_{G_J}(w)$-orbits in $X(w)$ labeled by normal flags of type $\{J(<r) \cup \{r\}\} \backslash \{n_2\}$ in $\mathcal{P}(V_2^*)$.*

The set $S^u(V, J, r)$ is defined for $r \in J \cup \{r\}$ and is in 1-1 correspondence with the $\mathrm{U}_{n_2}(q)$- orbits on normal chains of type $\{J(<r) \cup \{r\}\} \backslash \{n_2\}$. Let us examine the possible flags of such type in $\mathcal{P}(V_2^*)$. Let c be such a flag. Since V_2^* has a unitary structure, subspaces fall into two categories. They are either totally isotropic or not. The action of $\mathrm{U}_{n_2}(q)$ on V_2^* preserves this structure.

Definition 2.11 *If c is a flag of totally isotropic subspaces we will say that c is singular. If c contains a non-degenerate subspace we will say that c is a non-singular flag. Moreover we define the non-singular rank of c to be the dimension of the minimal non-degenerate subspace.*

If $\tau \in X(w)$ has rank r and corresponds to a non-singular flag c with non-singular rank r', then τ itself is said to have non-singular rank r'. Moreover $r' \in J \cup \{r\}$ since r' is the dimension of a subspace in the flag c corresponding to τ.

Suppose c is a flag of type $\{J(< r) \cup \{r\}\} \backslash \{n_2\}$ of non-singular rank r' with $1 \leq r' \leq r$. We have r', $r \in J \cup \{r\}$. Let $\widetilde{J} = \{k_1, k_2, \dots, k_s\}$ be the type of c. The element r' divides $\widetilde{J}$ into two subsets

$$\widetilde{J}_1 = \{\ k \mid k \in \widetilde{J}, k < r'\} \quad \text{and} \quad \widetilde{J}_2 = \{\ k \mid k \in \widetilde{J}, r' < k\}.$$

Write

$$c : 0 < V_{k_1} < \cdots < V_{r'} < \cdots < V_{k_s}.$$

We assign to c a pair (c_1, c_2) of shorter flags in the following way. The subspace $V_{r'}$ is the minimal non-degenerate subspace. We define

$$\begin{aligned} c_1 &: 0 < V_{k_1} < \cdots < V_{\max(\widetilde{J}_1)} \\ c_2 &: 0 < V_{\min(\widetilde{J}_2)} \cap V_{r'}^{\perp} < \cdots < V_{k_s} \cap V_{r'}^{\perp} \end{aligned}$$

The flag c_1 is a singular flag in the unitary space of dimension r'. The flag c_2 is a flag in the unitary space of dimension $n_2 - r'$.

Definition 2.12 *We define the following subsets of $S^u(V, J, r)$:*

1. *Let $S^{su}(V, J, r)$ be the subset of $S^u(V, J, r)$ labeled by singular flags.*
2. *Let $S^{nu}(V, J, r)$ be the subset of $S^u(V, J, r)$ labeled by non-singular flags.*
3. *Finally, let $S^{nu}_{r'}(V, J, r)$ be the subset of $S^{nu}(V, J, r)$ with non-singular rank r'.*

Observe that the set $S^{su}(V, J, r)$ is non-empty if and only if $r \leq n_2/2$, in which case it consists of a single member. Notice also that $S^{nu}_{r'}(V, J, r)$ is nonempty if and only if $r' \in J(< r)$ and $J(< r') \subseteq [n_2/2]$. We have the following structure of stabilizers of characters in these sets given by Ku.

Proposition 2.13 *([15], Remark 6.3.12)*

1. *For $\tau \in S^{su}(V, J, r)$ we have $T_{G_J}(\tau) = P^{+(n_1-r)}_{J_1} \times P^{n_2}_{\{J(<r) \cup \{r\}\} \backslash \{n_2\}}$*
2. *For $\tau \in S^{nu}_{r'}(V, J, r)$ we have $T_{G_J}(\tau) = P^{+(n_1-r)}_{J_1} \times P^{r'}_{J(<r')} \times T_{\mathrm{U}_{n_2-r'}(q)}(c_2)$*

where $J_1 = \{j - r \mid j \in J(> r)\}$ and c_2 is obtained as above from c which corresponds to τ.

2.6 The determinant map in the unitary linear module context

We would like to determine the determinant map on

$$G_J = P_J^{+n_1} \times \mathrm{U}_{n_2}(q),$$

as embedded in a larger unitary group. At this stage we consider the map

$$\begin{array}{rccc} \mathcal{D}: & P_J^{+n_1} \times \mathrm{U}_{n_2}(q) & \longrightarrow & F_{q^2}^* \\ & (A, B) & \mapsto & \det(A)^{i(1-q)} \det(B). \end{array}$$

for some positive integer i. The map $\mathcal{D}$ is constructed to be the restriction to G_J of the usual determinant map on $\mathrm{U}_n(q)$. The integer i depends on the embedding of G_J in $\mathrm{U}_n(q)$. For the rest of this subsection let $K = \ker(\mathcal{D})$

Observe the image of G_J under this map is all of $\mathbb{C}_{(q+1)}$ since $n_2 \neq 0$. We need to examine the map $\mathcal{D}$ restricted to the subgroup $T_{G_J}(\tau)$ and calculate $|T_{G_J}(\tau)K\backslash G_J|$.

Let $\tau \in S^{su}(V, J, r)$ so that

$$\begin{aligned} T_{G_J}(\tau) &= P_{J_1}^{+(n_1-r)} \times P_{J(<r)\cup\{r\}\backslash\{n_2\}}^{n_2} \\ &= P_{J_1}^{+(n_1-r)} \times \left(\left(P_{J(<r)}^{+r} \times \mathrm{U}_{n_2-2r}(q) \right) \ltimes U_r^{n_2} \right) \end{aligned}$$

where $U_r^{n_2}$ is the unipotent radical in the maximal parabolic subgroup $P_r^{n_2}$ in $\mathrm{U}_{n_2}(q)$. The map $\mathcal{D}$ restricted to $T_{G_J}(\tau)$ is trivial on the normal factor $U_r^{n_2}$ and

$$\begin{array}{rccc} \mathcal{D}: & P_{J_1}^{+(n_1-r)} \times P_{J(<r)}^{+r} \times \mathrm{U}_{n_2-2r}(q) & \longrightarrow & F_{q^2}^* \\ & (A, B, C) & \mapsto & (\det(A)^i \det(B)^{i+1})^{1-q} \det(C) \end{array}$$

where it is understood that if any of these dimensions are zero, we assume $\det(A) = 1$ for A in any group of dimension zero. The image of $\mathcal{D}$ on $T_{G_J}(\tau)$ depends on r, n_1, and n_2. There are three cases:

1. If $r = n_1 = n_2/2$, then the image of $T_{G_J}(\tau)$ under $\mathcal{D}$ is $\mathbb{C}_{(q+1)/\gcd(q+1,i+1)}$.
2. If $r < n_2/2$, then $\mathcal{D}(T_{G_J}(\tau)) = \mathbb{C}_{(q+1)}$.
3. If $r < n_1$ but $r = n_2/2$, then for any $\alpha \in F_{q^2}^*$, the diagonal matrix $A = (\alpha^{-1}, 1, \ldots, 1)$ is in $P_{J_1}^{+(n_1-r)}$ and the diagonal matrix $B = (\alpha, 1, \ldots, 1)$ is in $P_{J(<r)}^{+r}$. We have $\mathcal{D}(AB) = (\alpha^{-i}\alpha^{i+1})^{1-q} = \alpha^{1-q}$ and thus $\mathcal{D}(T_{G_J}(\tau))$ is all of $\mathbb{C}_{(q+1)}$.

Summarizing these results, we get the following proposition which is another step in computing the number of irreducible characters of P_J (in an appropriate p-block, of appropriate q-height) which split as desired upon restriction to $P_J \cap \mathrm{SU}_n(q)$:

Proposition 2.14 *For $\tau \in S^{su}(V, J, r)$*

$$|T_{G_J}(\tau)K\backslash G_J| = \begin{cases} \gcd(q+1, i+1), & \text{if } r = n_1 = n_2/2; \\ 1, & \text{otherwise.} \end{cases}$$

Now take $\tau \in S^{nu}(V, J, r)$ with non-singular rank $r' \neq 0$. We have

$$T_{G_J}(\tau) = P_{J_1}^{+(n_1-r)} \times P_{J(<r')}^{r'} \times T_{\mathrm{U}_{n_2-r'}(q)}(c_2)$$

where τ corresponds to c which corresponds to the pair (c_1, c_2) as above so that c_2 is a flag in the unitary space of dimension $n_2 - r'$. Then

$$\begin{array}{rccc} \mathcal{D}: & P_{J_1}^{+(n_1-r)} \times P_J^{r'}(< r') \times T_{\mathrm{U}_{n_2-r'}(q)}(c_2) & \longrightarrow & F_{q^2}^* \\ & (A, B, C) & \mapsto & \det(A)^{i(1-q)} \det(B)^{2i+1} \det(C). \end{array}$$

We take note of the restriction of $\mathcal{D}$ to the factor $P_{J(<r')}^{r'}$ in the stabilizer $T_{G_J}(\tau)$. For $A \in P_{J(<r')}^{r'}$, $\mathcal{D}(A) = \det(A)^{2i+1}$ since $-q \equiv 1 \mod (q+1)$. Notice also that $T_{\mathrm{U}_{n_2-r'}(q)}(c_2)$ is a stabilizer in a unitary group of smaller dimension than n_2. Later we will make use of an inductive argument.

2.7 Central Modules

In this section we first summarize Ku's results and then briefly examine a map which is related to the usual determinant map. At the end of this section we also examine how the useful cancellation applies to the central module case. We will be examining parabolic subgroups P_J in $\mathrm{U}_{n'}(q)$, where $n' \leq n$, as embedded in $\mathrm{U}_n(q)$. Let l be the maximal member of J. Write $Z_l = Z(U_l)$. Then the normal series for U_J terminates with $U_l \geq Z_l > 1$. Thus far the modules we have considered arise when $\chi \in \mathrm{Irr}(P_J)$ contains $Z(U_l)$ in its kernel. We will consider now the case when $\chi \in \mathrm{Irr}(P_J)$ does not contain $Z(U_l)$ in its kernel.

The parabolic group P_J has the following decomposition

$$P_J = \left(P_{J(<l)}^{+l} \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l$$

and acts on U_l by conjugation. This induces an action of the quotient group P_J/U_l on Z_l which is isomorphic as an abelian group to $M_{l,l}(q)$ which we denote by V. Indeed, since P_J is upper triangular we may write elements of V as matrices

$$\begin{pmatrix} a_{1,1} & a_{1,2} & & d_1 \\ a_{2,1} & & d_2 & \\ & & & \\ & & & -a_{1,2}^q \\ d_l & & -a_{2,1}^q & -a_{1,1}^q \end{pmatrix} \text{ in } M_{l,l}(q^2) \text{ where } d_i^q + d_i = 0.$$

Let V^l be the usual module for $\mathrm{GL}_l(q^2)$ then $V^l \otimes (V^l)^* \cong M_{l,l}(q^2)$ is a module for $P_{J(<l)}^{+l}$. Observe that $V \leq M_{l,l}(q^2)$. The quotient P_J/U_l acts on V as follows. For matrix $A \in P_{J(<l)}^{+l}$, $B \in \mathrm{U}_{n'-2l}(q)$, and $v \in V$, $(A, B) \cdot v = Av(\widetilde{A})^{-1}$ where $\widetilde{A}$ is as was defined in [1], i.e. if $A = (a_{i,j})$, then $\widetilde{A} = M((a_{j,i}^q))^{-1}M^{-1}$ where M is

the matrix with ones on the reverse diagonal. This induces an action on the subset $\mathrm{Irr}(V) \leq \mathrm{Irr}(V^l \otimes (V^l)^*) \cong \mathrm{Hom}(V^l, V^l)$ which is invariant on the rank of $\tau \in \mathrm{Irr}(V)$ which recall is the co-dimension of the kernel of τ viewed as a homomorphism from V^l to itself. Notice that the unitary factor acts trivially.

Case 1. We begin with the special case $n' = 2m$ and $J = \{m\}$ so that $P_J = \mathrm{GL}_m(q^2) \ltimes U_m$ and $U_m = Z_m \cong M_{m,m}(q)$.

Let $X = \mathrm{Irr}(V, r)$ the subset of characters in $\mathrm{Irr}(V)$ of rank r. Fix non-zero ϵ in the algebraic closure of F_q satisfying $\epsilon^q + \epsilon = 0$ and define the $m \times m$ matrix

$$x_r = (a_{i,j}) \text{ where } a_{i,j} = \begin{cases} \epsilon, & j - i = m - r; \\ 0, & \text{otherwise.} \end{cases}$$

As a matrix x_r has rank r.

Proposition 2.15 *([15], Chapter 7) The group $\mathrm{GL}_m(q^2)$ is transitive on X. The set $\{0, x_r \mid 1 \leq r \leq m\}$ is a complete set of representatives for the $\mathrm{GL}_m(q^2)$-orbits on $\mathrm{Irr}(V)$, where 0 denotes the zero matrix. Moreover*

$$T_{\mathrm{GL}_m(q^2)}(\tau_r) = T_{\mathrm{GL}_m(q^2)}(x_r) = \big(\mathrm{U}_r(q) \times \mathrm{GL}_{m-r}(q^2)\big) \ltimes M_{r,m-r}(q^2)$$

as in ([15], Lemma 7.3.1).

Case 2. Now consider the case where $U_l \neq Z_l$. Let $J = \{l\}$ so that

$$P_J = P_l \cong \big(\mathrm{GL}_l(q^2) \times \mathrm{U}_{n'-2l}(q)\big) \ltimes U_l$$

where $U_l/Z_l \cong M_{l,n'-2l}(q^2)$ and $Z_l \cong M_{l,l}(q)$. Let τ_r be identified with x_r. Since $\mathrm{U}_{n'-2l}(q)$ acts trivially on Z_l, by the first special case the set $\{1, \tau_r \mid 1 \leq r \leq l\}$ is a complete set of representatives for the P_J-orbits on $\mathrm{Irr}(Z_l)$, where 1 is the trivial character. Let

$$\mathrm{Irr}(U_l, \tau_r) = \{\phi \in \mathrm{Irr}(U_l) \mid \phi \text{ lies over } \tau_r\}.$$

If $\chi \in \mathrm{Irr}(P_l)$ does not contain Z_l in its kernel then χ lies over $\phi \in \mathrm{Irr}(U_l, \tau_r)$ where ϕ restricted to Z_l is a multiple of τ_r, for some $1 \leq r \leq l$. In this case, τ_r is not extendible to its stabilizer in P_l. However, it turns out that ϕ is extendible to $T_{P_l}(\phi)$.

Given $\phi \in \mathrm{Irr}(U_l, \tau_r)$, $\ker(\tau_r) \leq \ker(\phi)$. We may consider ϕ as a character of the quotient group $U_l/\ker(\tau_r)$. We may consider τ_r as a character of the quotient group $Z_l/\ker(\tau_r)$. The center Z_l is elementary abelian so $\mathrm{Irr}(Z_l) \cong \mathrm{Hom}_{F_p}(Z_l, F_p)$. Thus $Z_l/\ker(\tau_r)$ is cyclic of order p. The group $U_l/\ker(\phi)$ has an irreducible faithful representation and hence has cyclic center. Moreover, $Z(U_l/\ker(\phi))$ is a homomorphic image of $Z(U_l/\ker(\tau_r))$ which is elementary abelian ([15], p.101). Thus $Z(U_l/\ker(\phi))$ must have order p. We have

$$Z(U_l/\ker(\phi)) \geq Z_l \ker(\phi) \Big/ \ker(\phi) \cong Z_l \Big/ (\ker(\phi) \cap Z_l) = Z_l/\ker(\tau).$$

Hence

$$Z(U_l/\ker(\phi)) \cong Z_l/\ker(\tau).$$

Moreover

$$U_l/\ker(\phi) \Big/ Z\left(U_l/\ker(\phi)\right) \text{ is an elementary abelian } p\text{-group.}$$

Hence $U_l/\ker(\phi)$ is an extraspecial p-group. The ordinary character theory of such groups is well known. In summary, we have the following:

1. The order of $U_l/\ker(\phi)$ is p^{1+2a} for some integer a.
2. There are exactly p^{2a} linear characters, each corresponding to a character of the quotient
$$U_l/\ker(\phi) \Big/ Z\left(U_l/\ker(\phi)\right).$$
3. There are exactly $p-1$ non-linear characters each of dimension p^a. There is one of these characters χ for each non-trivial irreducible character θ of $Z(U_l/\ker(\phi))$ with character values given by $\chi(x) = p^a\theta(x)$ for x in $Z(U_l/\ker(\phi))$ and $\chi(x) = 0$ for x not in $Z(U_l/\ker(\phi))$.

Considered as a character of $U_l/\ker(\tau_r)$, ϕ is uniquely determined by its kernel. We summarize the properties of ϕ.

Proposition 2.16 *([15], Lemma 7.2.4) Let $1 \leq r \leq l$ and $\phi \in \mathrm{Irr}(U_l, \tau_r)$.*

1. *ϕ is extendible to $T_{P_l}(\phi)$*
2. *$\phi(1) = q^{r(n-2l)}$*
3. *If $r = l$, then $\mathrm{Irr}(U_l, \tau_l)$ contains a unique member and*
$$T_{P_l}(\phi) = T_{P_l}(\tau_l) \cong \left(\mathrm{U}_l(q) \times \mathrm{GL}_{n'-2l}(q^2)\right) \ltimes U_l.$$
4. *If $1 \leq r < l$, then ϕ is uniquely determined by its kernel.*
$$T_{P_l}(\phi) = T_{L_l}(\phi) \ltimes U_l = (T_{L_l}(\tau_r) \cap T_{L_l}(\ker(\phi))) \ltimes U_l.$$

For a non-negative integer k, let V^k be the natural module for $\mathrm{GL}_k(q^2)$. Let $R \leq V^l$ be a subspace of codimension r stabilized by $T_{L_l}(\tau_r)$. Then

$$V^{l-r} \otimes (V^{n'-2l})^*$$

is a module for $\mathrm{GL}_{l-r}(q^2) \times \mathrm{U}_{n'-2l}(q)$. Moreover, there is a $1-1$ correspondence

$$\mathrm{Irr}(U_l, \tau_r) \longleftrightarrow \mathrm{Irr}(V^{l-r} \otimes (V^{n'-2l})^*).$$

Case 3. In order to count characters of P_J that do not contain Z_l in their kernel, we observe that

$$\begin{aligned} k_d^1(P_J, Z_l, \rho, \det, j) &= \sum_{\tau \in Z_l/P_J} k_d(P_J, \tau, \rho, \det, j) \\ &= \sum_{\tau \in Z_l/P_J} \sum_{\phi \in \mathrm{Irr}(U_l, \tau)} k_d(P_J, \phi, \rho, \det, j) \\ &= \sum_{\tau \in Z_l/P_J} \sum_{\phi \in \mathrm{Irr}(U_l, \tau)} k_{d-d'}(T_{P_J}(\phi)/U_l, \rho, \det, j'). \end{aligned}$$

where $d' = r(n' - 2l) - d''$ for d'' equal to the exponent of q in $|T_{P_J}(\phi) \backslash P_J|$ and j' is the least positive integer such that

$$j \text{ divides } j' \cdot \mid T_{P_J}(\phi) \ker(\det) \backslash P_J \mid .$$

We use Proposition 2.16 to describe P_J orbits on $X = \mathrm{Irr}(V, r)$ for the general case where J is any subset in I with maximal element l.

Definition 2.17 *For fixed r, let $\mathcal{K} = T_{\mathrm{GL}_l(q^2)}(\tau_r)$.*

Let w be zero if $r = l$, or if $r < l$ a complement in V^l to the r dimensional subspace stabilized by P_r^{+l}. We have the structure of $\mathcal{K}$ given above by Proposition 2.15.

Proposition 2.18 *There is a 1-1 correspondence between the P_J-orbits on X and the $\mathcal{K}$-orbits on the set of chains of type $J(< l)$ in $\mathcal{P}(V^l)$. If τ corresponds to c then up to conjugation*

$$T_{P^{+l}_{J(<l)}}(\tau) = T_{\mathcal{K}}(c).$$

Let $r = l$ so that $\mathcal{K} = T_{\mathrm{GL}_l(q^2)}(\tau_l) = \mathrm{U}_l(q)$.

Definition 2.19 *Let $S^z(V, J, l)$ denote the P_J-orbits in X labeled by a normal chain in $\mathcal{P}(V^l)$ of type $J(< l)$.*

Let $r < l$ so that w is a complement in V^l to the r dimensional subspace stabilized by P_r^{+l}. We can assume that $\mathcal{K} \leq T_{\mathrm{GL}_l(q^2)}(w)$. Then $\mathcal{K}$ is transitive on complements of w. Moreover if $\tau \in X$ corresponds to c of type $J(< l)$ with

$$c : 0 < V_1 < \cdots < V_i < \cdots < V_s$$

where V_i is a complement to w then we may assign to c a pair (c_1, c_2) of shorter chains much as we did in the previous section. As $(V_i)^\perp = w$ we may define

$$\begin{aligned} c_1 &: 0 < V_1 < \cdots < V_{i-1} \\ c_2 &: 0 < (V_{i+1} \cap w) < \cdots < (V_s \cap w). \end{aligned}$$

Notice that c_1 is a chain of type $J(<r)$ in $\mathcal{P}(V^r)$ and c_2 of of type $\{j-r|j \in J(>r)\}$ in $\mathcal{P}(V^{l-r})$. We have the stabilizer of c in $\mathcal{K}$ given by

$$T_{\mathcal{K}}(c) = T_{\mathrm{U}_r(q)}(c_1) \times T_{\mathrm{GL}_{l-r}(q^2)}(c_2).$$

With this identification of c with the pair (c_1, c_2), we make the following definition.

Definition 2.20 *Let $S^z(V,J,r)$ denote the P_J-orbits in X labeled by a normal chain c_1 of type $J(<r)$ in $\mathcal{P}(V^r)$.*

By construction $S^z(V,J,r)$ is in 1-1 correspondence with the $\mathrm{U}_r(q)$-orbits on the set of chains of type $J(<r)$ in $\mathcal{P}(V^r)$.

Definition 2.21 *We define the following subsets of $S^z(V,J,r)$.*

1. *Let $S^z_r(V,J,r)$ be the subset labeled by a singular normal chain c_1 of type $J(<r)$ in $\mathcal{P}(V^r)$.*
2. *For $r'<r$ let $S^z_{r'}(V,J,r)$ be the subset labeled by a normal chain c_1 of type $J(<r)$ in $\mathcal{P}(V^r)$ with non-singular rank r'.*

We may now assign to c_1 a pair of even shorter chains based on the dimension of the minimal non-degenerate subspace in c_1. In the manner of the previous section c_1 corresponds to the pair (c_{11}, c_{12}) where c_{11} is a totally isotropic chain in $\mathcal{P}(V^{r'})$ and c_{12} is a normal chain in $\mathcal{P}(V^{r-r'})$.

In summary for J with maximal member l, $r \in J$ and $\tau \in X = \mathrm{Irr}(V,r)$ corresponding to (c_1, c_2) where c_1 corresponds to (c_{11}, c_{12}) we have

$$\begin{aligned} T_{P_J}(\tau) = & \left(T_{P^{+l}_{J(<l)}}(\tau) \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l \\ & \left(T_{\mathrm{U}_r(q)}(c_1) \times T_{\mathrm{GL}_{l-r}(q^2)}(c_2) \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l \\ & \left(T_{\mathrm{U}_{r'}(q)}(c_{11}) \times T_{\mathrm{U}_{r-r'}(q)}(c_{12}) \times T_{\mathrm{GL}_{l-r}(q^2)}(c_2) \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l \\ & \left(P^{r'}_{J(<r')} \times T_{\mathrm{U}_{r-r'}(q)}(c_{12}) \times T_{\mathrm{GL}_{l-r}(q^2)}(c_2) \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l. \end{aligned}$$

where if $r=l$ then we take $c_2=0$. Now let $\phi \in \mathrm{Irr}(U_l, \tau_r)$ correspond to $\psi \in \mathrm{Irr}(V^{l-r} \otimes (V^{n'-2l})^*)$ and let

$$D = T_{\mathrm{GL}_{l-r}(q^2)}(c_2) \times \mathrm{U}_{n'-2l}(q).$$

Then

$$T_{P_J}(\phi) = (T_{P_J}(\tau) \times T_D(\psi)) \ltimes U_l.$$

2.8 The determinant map in the central module context

We would like to determine the determinant map on quotients of the form $P_J/U_l \leq \mathrm{U}_{n'}(q)$, as embedded in the larger unitary group $\mathrm{U}_n(q)$. Here P_J is a parabolic subgroup of $\mathrm{U}_{n'}(q)$. We have

$$P_J/U_l \cong P^{+l}_{J(<l)} \times \mathrm{U}_{n'-2l}(q).$$

At this stage we consider the same map $\mathcal{D}$ from section 2.6 on unitary linear modules.

$$\begin{array}{rccc} \mathcal{D}: & P^{+l}_{J(<l)} \times \mathrm{U}_{n'-2l}(q) & \longrightarrow & F^*_{q^2} \\ & (A,B) & \mapsto & \det(A)^{i(1-q)}\det(B), \end{array}$$

for some positive integer i. The map $\mathcal{D}$ is constructed to be the restriction to P_J/U_l of the usual determinant map on $\mathrm{U}_n(q)$. The integer i depends on the embedding of P_J/U_l in $\mathrm{U}_n(q)$.

Observe the image of P_J/U_l under this map is all of $\mathbb{C}_{(q+1)}$ if $n' - 2l \neq 0$. On the other hand if $n' = 2l$ then the image is $\mathbb{C}_{(q+1)/\gcd(q+1,i)}$.

We briefly remark on the restriction of the map $\mathcal{D}$ to the factor $P^{r'}_{J(<r')}$ in the stabilizer $T_{P_J}(\tau)$. For $A \in P^{r'}_{J(<r')}$, $\mathcal{D}(A) = \det(A)^{2i}$ since $-q \equiv 1 \mod (q+1)$.

Remark: A version of the useful cancellation discussed in section 2.2 applies to the central modules in the following way: Let G be the subgroup of $\mathrm{GL}_l(q^2) \times \mathrm{GL}_l(q^2)$ defined

$$G = \{(A, \widetilde{A}) | A \in \mathrm{GL}_l(q^2)\}.$$

Let V^l be the natural module for $\mathrm{GL}_l(q^2)$. Set $\mathbf{V} = V^l \otimes (V^l)^*$. Let G act on $\mathbf{V}$ via $A \cdot v = Av\widetilde{A}^{-1}$. In keeping with the notation used in section 2.1, $l = n_1 = n_2$ and G_2 is an isomorphic copy of G_1. As an F_{q^2}-vector space, recall a basis for $\mathbf{V}$ was given by $\{E_{i,j}\}$. View F_{q^2} as an extension of F_q. Let ϑ be a root of the irreducible polynomial

$$x^2 - (\vartheta + \vartheta^q) + \vartheta^{q+1} \text{ in } F_q[x]$$

so that $F_{q^2} = F_q(\vartheta)$. Let V be the F_q-subspace of $\mathbf{V}$ with basis given by

$$\{E_{i,j} - E_{l-j+1,l-i+1}\} \bigcup \{\vartheta E_{i,j} - \vartheta^q E_{l-j+1,l-i+1}\}.$$

The subspace V, and hence $\mathrm{Irr}(V)$, is closed under the action of G. Let $X = \mathrm{Irr}(V, r)$. Then X is a subset of $\mathrm{Irr}(\mathbf{V}, r)$. For $J \subseteq [l-1]$, define the subgroup

$$G_J = \{(A, \widetilde{A}) | A \in P^{+l}_J\} \leq G.$$

Let $H = G \ltimes V$ and $H_J = G_J \ltimes V$. In this case for $(A, \widetilde{A}) \in G_J$,

$$\mathcal{D}(A, \widetilde{A}) = \det(A)^{i(1-q)}$$

for some integer i.

Then $Y = \{y \leq V^l | \mathrm{codim}(y) = r\}$ is a transitive H-set. If J has maximal member l then

$$P_J/U_l \cong G_{J(<l)} \times \mathrm{U}_{n'-2l}(q) \text{ and } Z_l \cong V.$$

If $r = l$ then $X(0) = X$ and $T_{G_J}(0) = G_J$. Now suppose that $r < l$. Set w to be a complement to the r-dimensional subspace R of V^l stabilized by P_r^{+l}. The vector space $V^l = R \oplus w$. Take $\tau \in X(w)$ so that viewed as a linear transformation of V^l, τ has kernel equal to w. Now even if r is not a member of $J(< l)$ we have

$$T_{G_J}(w) = M_{r,l-r}(q^2) \rtimes \left(P^{+r}_{J(<r)} \times P^{+(l-r)}_{J_1}\right)$$

where $J_1 = \{j - r | j \in J(> r)\}$. For $y \in Y$, $y = Aw$ for some $A \in \mathrm{GL}_l(q^2)$, hence $T_{G_J}(y)$ is $\mathrm{GL}_l(q^2)$-conjugate to $T_{G_J}(w)$.

The main difference in applying the useful cancellation to the central module context is in the definition of a G-stable function. We define

$$\begin{aligned} f : \Delta(P) \times Y &\longrightarrow \mathbb{Z} \\ (c_J, y) &\mapsto k_d(P_{J\cup\{l\}}, X(y), \rho, \mathcal{D}, j). \end{aligned}$$

Notice that $P_{J\cup\{l\}}$ is not H_J. Also notice that $l - r \neq 0$ in this case so that

$$|T_{P_J}(y) \ker(\mathcal{D})\backslash P_J| = 1 \text{ holds.}$$

With these changes in the proof of Proposition 2.6 the useful cancellation applies to the the central module case.

Proposition 2.22 *Let $Z \leq Z(G)$ and $\rho \in Irr(Z)$. Fix $1 \leq r \leq l$. Let $V = Z_l$ and let $X = \mathrm{Irr}(V, r)$. Let w be a complement in V^l to the r-dimensional subspace stabilized by P_r^{+l}, in its action on the natural module for $\mathrm{GL}_l(q^2)$. Let $X(w) = \{\tau \in X \mid \ker(\tau) = w\}$. Let $\mathcal{D}$ be the map on P_J defined above. For $d \geq 0$ the following holds*

1. *If $r = l$, then $Y = 0$ and $X = X(0)$.*
2. *If $r < l$, then*

$$\sum_{J\subseteq[l-1]} (-1)^{|J|} k_d(P_{J\cup\{l\}}, X, \rho, \mathcal{D}, j) = \sum_{\substack{J\subseteq[l-1] \\ r\in J}} (-1)^{|J|} k_d(P_{J\cup\{l\}}, X(w), \rho, \mathcal{D}, j).$$

Representatives of $T_{G_J}(w)$-orbits in $X(w)$ are given by chains of type $J(< l) \cup \{r\}$ in $\mathcal{P}(V^l)$ where c corresponds to (c_1, c_2) as described earlier in this section.

Having examined the orbits, stabilizers, and maps $\mathcal{D}$ at the level of H_J in the linear cases (both general and unitary) and $P_J^{n'}$ in the central case, we are ready to proceed up to the level of P_J^n. We will do so in the following section where we begin a systematic codification of the alternating sum that occurs on the left hand side of 1b.

3 Counting characters of parabolic subgroups not trivial on the unipotent radical

For fixed $J \subset I$ write $J = \{j_1, j_2, \ldots, j_s\}$ in increasing order. Then U_J has the following normal series:

$$U_J = U_{J(\geq j_1)} > U_{J(\geq j_2)} > \cdots > U_{J(\geq j_s)} = U_{j_s} \geq Z(U_{j_s}) > 1 \tag{4}$$

The quotient groups in this series are abelian. These are the modules described as general linear, unitary linear, and central modules in the previous section where recall we examined their H_J orbits. If $\chi \in \mathrm{Irr}(P_J)$ does not contain U_J in its kernel then there exists a term in the above series which is contained in the kernel of χ, but the previous term is not in the kernel of χ.

Recall $V(j_i, j_{i+1})$ denotes the quotient group $U_{J(\geq j_i)}/U_{J(\geq j_{i+1})}$. We note that in the following definition, *1.* is a slight modification of definition 1.2 but that *2.* has already been defined and is only restated for clarity.

Definition 3.1 *For $\rho \in \mathrm{Irr}(Z)$:*

1. *Let $k_d^1(P_J, V(j_i, j_{i+1}), \rho, \det, j)$ be the number of $\chi \in \mathrm{Irr}(P_J)$ which are trivial on $U_{J(\geq j_{i+1})}$ but not trivial on $U_{J(\geq j_i)}$, lie over ρ, and upon restriction to* $\ker(\det)$ *have j' irreducible constituents where $j \mid j'$.*

2. *Let $k_d^1(P_J, U_{j_s}, \rho, \det, j)$ be the number of $\chi \in \mathrm{Irr}(P_J)$ which are not trivial on U_{j_s}, lie over ρ, and upon restriction to* $\ker(\det)$ *have j' irreducible constituents where $j \mid j'$.*

Case 1. If $\chi \in \mathrm{Irr}(P_J)$ is trivial on $U_{J(\geq j_{i+1})}$ but not trivial on $U_{J(\geq j_i)}$, then we may consider χ as a character of

$$\overline{P_J} = P_J/U_{J(\geq j_{i+1})} \cong P^{+j_{i+1}}_{J(\leq j_i)} \times L_{J'}$$

where $J' = \{j - j_{i+1} | j \in J(> j_{i+1})\}$ and $L_{J'}$ is isomorphic to a Levi subgroup in $\mathrm{U}_{n-2j_{i+1}}(q)$. Then

$$P^{+j_{i+1}}_{J(\leq j_i)} \cong \left(P^{+j_i}_{J(<j_i)} \times \mathrm{GL}_{j_{i+1}-j_i}(q)\right) \ltimes V(j_i, j_{i+1}), \tag{5}$$

as in subsection 2.3 of section 2, on general linear modules, where $P^{+j_{i+1}}_{J(\leq j_i)}$ plays the role of H_J. Hence χ is of the following form:

$$\chi = \chi'(\tilde{\tau}\psi)^{P^{+j_{i+1}}_{J(\leq j_i)}}$$

where $\tau \in \mathrm{Irr}(V(j_i, j_{i+1}))$ is linear and hence extendible to $\tilde{\tau} \in \mathrm{Irr}(T)$ and $\psi \in \mathrm{Irr}(T/V(j_i, j_{i+1}))$ where T is the stabilizer of τ in $P^{+j_{i+1}}_{J(\leq j_i)}$ and χ' is an irreducible character of the factor $L_{J'}$. It follows that χ corresponds to τ and that

$$k_d^1(P_J, V(j_i, j_{i+1}), \rho, \det, j) = \sum_{\tau} k_d(P_J, \tau, \rho, \det, j)$$

where this sum is taken over representatives τ of P_J-orbits in $\mathrm{Irr}(V(j_i, j_{i+1}))$.

Case 2. Suppose that $\chi \in \mathrm{Irr}(P_J)$ is not trivial on U_{j_s}. There are two possibilities: χ is trivial on $Z(U_{j_s}) = Z_{j_s}$ or not.

Case 2a. If χ is trivial on Z_{j_s}, then we may consider χ as a character of

$$P_J/Z_{j_s} \cong \left(P^{+j_s}_{J(<j_s)} \times \mathrm{U}_{n-2j_s}(q)\right) \ltimes (U_{j_s}/Z_{j_s}), \tag{6}$$

as in subsection 2.5 of section 2, on unitary linear modules, where P_J/Z_{j_s} plays the role of H_J. Hence χ is of the following form:

$$\chi = (\tilde{\tau}\psi)^{P_J/Z_{j_s}}$$

where $\tau \in \mathrm{Irr}(U_{j_s}/Z_{j_s})$ linear and hence extendible to $\tilde{\tau} \in \mathrm{Irr}(T)$ and $\psi \in \mathrm{Irr}(T/(U_{j_s}/Z_{j_s}))$ where T is the stabilizer of τ in P_J/Z_{j_s}. It follows that χ corresponds to an irreducible character of the quotient U_{j_s}/Z_{j_s}, and that

$$k^1_d(P_J, U_{j_s}/Z_{j_s}, \rho, \det, j) = \sum_{\tau} k_d(P_J, \tau, \rho, \det, j)$$

where this sum is taken over nontrivial representatives τ of P_J-orbits in $\mathrm{Irr}(U_{j_s}/Z_{j_s})$.

Case 2b. Suppose χ is not trivial on Z_{j_s}. As discussed in subsection 2.7 of section 2, χ corresponds to $\phi \in \mathrm{Irr}(U_{j_s})$ where ϕ lies over a non-trivial character $\tau_r \in \mathrm{Irr}(Z_{j_s})$. Set $N = \ker(\phi)$ then $K = N \cap Z_{j_s}$ is non-trivial. Then U_{j_s}/N is an extra special p-group and hence the non-linear $\phi \in \mathrm{Irr}(U_{j_s}/N)$ is extendible to $\tilde{\phi} \in \mathrm{Irr}(T)$ where T is the stabilizer of ϕ in P_J/N. Thus χ is of the form:

$$\chi = (\tilde{\phi}\psi)^{P_J}$$

where $\phi \in \mathrm{Irr}(U_{j_s}/N)$ is the unique character whose restriction to Z_{j_s}/K is a multiple of non-trivial $\tau \in \mathrm{Irr}(Z_{j_s}/K)$. The character ψ is the lift to T of an irreducible character of $T/(U_{j_s}/N)$. The character ϕ lifts to an irreducible character in $\mathrm{Irr}(U_{j_s})$ and τ lifts to an irreducible character in $\mathrm{Irr}(Z_{j_s})$. It follows that χ corresponds to an irreducible character of Z_{j_s} and that

$$k^1_d(P_J, Z_{j_s}, \rho, \det, j) = \sum_{\tau} \sum_{\substack{\phi \\ \phi \in \mathrm{Irr}(U_{j_s}, \tau)}} k_d(P_J, \phi, \rho, \det, j)$$

where this sum is taken over nontrivial representatives τ of P_J orbits in $\mathrm{Irr}(Z_{j_s})$.

We have the following decomposition:

$$k^1_d(P_J, U_J, \rho, \det, j) = \sum_{i=1}^{s-1} k^1_d(P_J, V(j_i, j_{i+1}), \rho, \det, j) + k^1_d(P_J, U_{j_s}, \rho, \det, j). \tag{7}$$

For fixed J containing adjacent elements l and l', let $V = V(l, l') = U_{J(\geq l)}/U_{J(\geq l')} \cong V_1 \otimes V_2$, where we recall that V_1 is the natural module for $\mathrm{GL}_l(q^2)$ and V_2 is the dual of the natural module for $\mathrm{GL}_{l'-l}(q^2)$. Then

$$\overline{P}_J = P_J/U_{J(\geq l')} \text{ contains a submodule isomorphic to } V \text{ (see 5).}$$

The useful cancellation from section 2 applies in this situation. We will sum over all $J \subseteq I$ of the form $J = J' \cup J''$ where $J' \subseteq [l-1]$ varies and $J'' \subset I$ is fixed with minimal member l'. We will only be concerned with calculating $T_{H_J}(\tau)$ where τ has rank $r \in J$ and $\ker(\tau)$ is a complement w in V_1 to the r-dimensional subspace stabilized by P_r^{+l}. Then, for $r \in J$ and $\tau \in \mathrm{Irr}(V, r)$ with $\ker(\tau) = w$

$$k_d(P_J, \tau, \rho, \det, j) = k_d(\overline{P}_J/V, \tau, \rho, \det, j) = k_{d-d'}(T_{\overline{P}_J/V}(\tau), \rho, \det, j') \tag{8}$$

where d' is the power of q in the index of $T_{\overline{P}_J/V}(\tau)$ in $\overline{P}_J/V$and j' is the smallest positive integer such that

$$j \mid j' \cdot \left| T_{\overline{P}_J/V}(\tau) \cdot \ker(\det) \backslash (\overline{P}_J/V) \right| .$$

It turns out that $T_{\overline{P}_J/V}(\tau)$ contains a subgroup which itself contains a submodule isomorphic to a general linear module and hence we may further expand 8.

Let J have maximal element l. Let $V = U_l/Z_l \cong V_1 \otimes V_2$, where recall V_1 is the natural module for $\mathrm{GL}_l(q^2)$ as above and V_2 is the dual of the natural module for $\mathrm{U}_{n-2l}(q)$. Then

$$\overline{P}_J = P_J/Z_l \text{ contains a submodule isomorphic to } V \text{ (see 6).}$$

As above, the useful cancellation from section 2 applies to this situation. We will when sum over all $J \subseteq I$ with maximal element l. We will only be concerned with calculating $T_{H_J}(\tau)$ where τ has rank $r \in J$ and $\ker(\tau)$ is a complement w in V_1 to the r-dimensional subspace stabilized by P_r^{+l}. Then, for $r \in J$ and $\tau \in \mathrm{Irr}(V, r)$ with $\ker(\tau) = w$,

$$k_d(P_J, \tau, \rho, \det, j) = k_d(\overline{P}_J/V, \tau, \rho, \det, j) = k_{d-d'}(T_{\overline{P}_J/V}(\tau), \rho, \det, j') \tag{9}$$

where d' and j' are as given above in 8. It turns out that when τ corresponds to a singular chain in the unitary vector space V_2^*, $T_{\overline{P}_J/V}(\tau)$ contains a subgroup which itself contains submodule isomorphic to a (unitary) quotient module and hence we may further expand 9.

The main goal in this section is to unravel 7 via 8 and 9. Ku has introduced two sets of triples E and F with related objects which codify this unraveling for the unitary case but without regard to the determinant map or any splitting. For an $e \in E$ or an $f \in F$ we will present Ku's objects including a length, a parity, a group, and a normal subgroup. For our purposes we will also define a map related to the determinant and an integer related to splitting. We remark that while the definitions of these two new objects are very natural extensions of Ku's existing objects, the computations involved in producing their definitions are far from trivial. The set E will codify parabolic characters corresponding to internal general linear modules, i.e. those counted in the alternating sum

$$\sum_{J\subseteq I}(-1)^{|J|}\sum_{i=1}^{s-1}k_d^1(P_J,V(j_i,j_{i+1}),\rho,\det,j),$$

whereas the set F will codify the parabolic characters corresponding to internal unitary linear or central modules, i.e. those characters counted in the alternating sum

$$\sum_{J\subseteq I}(-1)^{|J|}k_d^1(P_J,U_{j_s},\rho,\det,j).$$

Before proceeding to the rather technical definitions, we present the idea. Fix nonempty $J\subseteq I$.

- For each pair of adjacent members l and l' of J, we will define an initial triple e so that its associated group

  $$P(e)\cong P_J/U_{l'}\cong\Big(\Big(P^{+l}_{J(<l)}\times\mathrm{GL}_{l'-l}(q^2)\Big)\ltimes V(l,l')\Big)\times L_{J'}$$

  where $J'=\{j-l'|j\in J(>l')\}$ and $L_{J'}$ is a Levi subgroup in $\mathrm{U}_{n-2l'}(q)$. Moreover, for $\tau\in\mathrm{Irr}(V(l,l'))$ of rank r where $r\in J$ and $\ker(\tau)$ is a complement w to the r-dimensional subspace stabilized by P_r^{+l}, we will define a subsequent triple e' related to e such that

  $$T_{P(e)}(\tau)=P(e')\ltimes V(l,l').$$

  If $P(e')$ itself contains a non trivial general linear module we will define a further triple e'' related to e' in a similar way. In this way J, l, and l' give rise to a sequence e, e', e'',... of elements in E.

- Now let l be the maximal member of J. We will define an initial triple f so that its associated group

  $$P(f)\cong P_J\cong\Big(P^{+l}_{J(<l)}\times\mathrm{U}_{n-2l}(q)\Big)\ltimes U_l$$

  Moreover, for $\tau\in\mathrm{Irr}(U_l/Z_l)$ of rank r where $r\in J$, $\ker(\tau)$ is a complement w to the r-dimensional subspace stabilized by P_r^{+l}, and τ corresponds to a singular flag in a unitary space of suitable dimention, we will define a subsequent triple f' related to f such that

  $$T_{P(f)}(\tau)=P(f')\ltimes U_l.$$

  If $P(f')$ itself contains a non trivial unitary linear module with an irreducible singular character (i.e. corresponds to a singular flag in a unitary space of suitable dimension) we will define a further triple f'' related to f' in a similar way. In this way J and l give rise to a sequence f, f', f'',... of elements in F.

In this fashion, we will use the elements of E and F to reformulate 1b in a systematic way by unraveling 8 and 9 which leads to a second reduction of 1b.

The notation used in the following sections is primarily due to Ku. In particular he defines the following: E, $P(e)$, $V(e)$, $l(e)$, $|e|$, and $d(e)$ used in section 3.1; F, $P(f)$, $U(f)$, $Z(f)$, $V(f)$, $l(f)$, $|f|$, and $d(f)$ used in section 3.2; $S^u(f)$, $S^{su}(f)$, $S^{nu}(f)$, $S^z(f)$ used in section 3.3.

3.1 The elements of E and their related objects

Let e be an ordered triple $(J, C, (l, l'))$, where either $e = (\emptyset, \emptyset, \emptyset)$ or J, C, and (l, l') satisfy the following conditions:

1. $J = \{j_1, j_2, \ldots, j_r\}$ is a subset of $I = [m]$. We will assume that J is enumerated in increasing order.
2. (l, l') is a pair of consecutive members of J.
3. $C = \{l_1, l_2, \ldots, l_s\}$ is a subset of $I = [m]$ also enumerated in increasing order. The sequence C must be convex. By this we mean that the related sequence $\partial C = (l_1, l_2 - l_1, \ldots, l_s - l_{s-1})$ is non-increasing. We require that the members of this related sequence appear in J. We further require that C and (l, l') are related via $l = l_1 < l_2 < \ldots < l_s \leq l'$.

Let E be the set of all such triples e. The length of e is $l(e) = |C| = s$ and the parity of e is $|e| = |J|$. Before we proceed to the rest of the objects related to $e \in E$, we make some observations.

Definition 3.2 *Given $C = \{l_1, l_2, \ldots, l_s\}$, we define $\partial l_1 = l_1$ and $\partial l_i = l_i - l_{i-1}$ for $2 \leq i \leq s$ so that ∂C is the partition $(\partial l_1, \partial l_2, \ldots, \partial l_s)$.*

Observe that a convex sequence C corresponds to a unique partition $\partial C = (\partial l_1, \partial l_2, \ldots, \partial l_s) \vdash l_s$. Notice that the ∂l_i's, where $l = \partial l_1 = l_1$, and l' are contained in J. Moreover, l and l' are consecutive members of J. Thus the sequence $\partial l_s \leq \partial l_{s-1} \leq \cdots \leq \partial l_2 \leq \partial l_1 = l < l'$ divides J into a collection of $s+1$ disjoint subsets

$$\{j \in J | j < \partial l_s\}, \quad \cdots, \quad \{j \in J | \partial l_2 < j < \partial l_1\}, \quad \{j \in J | l' < j\}.$$

Definition 3.3 *We define*

1. *$J_s = \{j \mid j \in J$ and $j < \partial l_s\}$,*
2. *$J_i = \{j - \partial l_{i+1} \mid j \in J$ and $\partial l_{i+1} < j < \partial l_i\}$ for $1 \leq i < s$, and*
3. *$J_0 = \{j - l' \mid j \in J$ and $l' < j\}$.*

Notice that we have the containment $J_s \subset [\partial l_s - 1]$, $J_i \subset [\partial l_i - \partial l_{i+1} - 1]$, and $J_0 \subset [m - l']$. We are now ready to define the groups related to the triple $e = (J, C, t)$.

Definition 3.4 *For $e \in E$ we define the following objects.*

1. *Let*
$$P(e) = L^{n_0(e)}_{J_0(e)} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times P^{+n_{s-1}(e)}_{J_{s-1}(e)} \times P^{+n_s(e)}_{J_s(e)}$$
where

   (a) $n_0(e) = n - 2l'$, $J_0(e) = J_0$

   (b) $n_i(e) = \partial l_i - \partial l_{i+1}$ *and* $J_i(e) = J_i$ *for* $1 \leq i < s$

   (c)
$$n_s(e) = l' - l_{s-1} \text{ and } J_s(e) = \begin{cases} J_s, & \text{if } l_s = l'; \\ J_s \cup \{\partial l_s\}, & \text{if } l_s < l'. \end{cases}$$

   Notice that the first factor is a Levi subgroup of a unitary group whereas the remaining s factors are parabolic subgroups of general linear groups.

2. *We define an abelian normal subgroup of $P(e)$ in the following way. If $l_s < l'$ then the group $P^{+n_s(e)}_{J_s(e)}$ is contained in the maximal parabolic $P^{+n_s(e)}_{\partial l_s}$ which has unipotent radical $U^{+n_s(e)}_{\partial l_s}$, a normal subgroup of $P^{+n_s(e)}_{J_s(e)}$ and hence of $P(e)$. Set*
$$V(e) = \begin{cases} U^{+n_s(e)}_{\partial l_s}, & \text{if } l_s < l'; \\ 1, & \text{if } l_s = l'. \end{cases}$$
Observe that
$$P^{+n_s(e)}_{J_s(e)} \cong \left(P^{+\partial l_s}_{J_s} \times \mathrm{GL}_{l'-l_s}(q^2)\right) \ltimes V(e). \tag{10}$$
Moreover $V(e)$ is a general linear module for $P^{+\partial l_s}_{J_s} \times \mathrm{GL}_{l'-l_s}(q^2)$ as discussed in Section 2.3.

3. *We define*
$$d(e) = 2\sum_{i=1}^{s-1}\left(\binom{\partial l_i}{2} - \binom{\partial l_i - l_{i+1}}{2}\right).$$
Notice that $d(e)$ depends only on the sequence C.

4. *Finally we define a map $\phi_e : P(e) \to F_{q^2}$. For $g \in P(e)$, write $g = A_0 A_1 \cdots A_{s-1} A_s$ where $A_0 \in L^{n_0(e)}_{J_0(e)}$ and $A_i \in P^{+n_i(e)}_{J_i(e)}$. We have a decomposition of $A_s \in P^{+n_s(e)}_{J_s(e)}$. Write $A_s \equiv A_{s,1} A_{s,2} \mathrm{mod}(V(e))$ where $A_{s,1} \in P^{+\partial l_s}_{J_s}$ and $A_{s,2} \in \mathrm{GL}_{l'-l_s}(q^2)$. Let*
$$\phi_e(g) = \det A_0 \left[\left(\prod_{i=1}^{s-1} (\det A_i)^i\right) (\det A_{s,1})^s \det A_{s,2}\right]^{1-q} \tag{11}$$

where det *denotes the usual determinant map. Notice* $\phi_e(P(e)) \leq \mathbb{C}_{q+1}$*.*

In the context of general linear modules, the map $\mathcal{D}$ *defined in Section 2.3 on* $P^{+n_s(e)}_{J_s(e)}$ *corresponds to* $i = s$ *and is the restriction of* ϕ_e *to* $P^{+n_s(e)}_{J_s(e)}$*.*

An example of E: Fix J with adjacent members l and l'. Set the initial $e = (J, \{l\}, (l, l'))$. Then $\partial C = (l)$

$$\begin{aligned} P(e) \cong P_J/U_{J(<l')} &\cong L^{n-2l'}_{J_0} \times P^{+l'}_{J(\leq l)} \\ &\cong L^{n-2l'}_{J_0} \times \left(P^{+l}_{J(<l)} \times \mathrm{GL}_{l'-l}(q^2)\right) \ltimes V(e) \end{aligned}$$

and $V(e) \cong V(l, l') \cong M_{l,l'-l}(q^2)$. Under this isomorphism the map ϕ_e is the determinant map on $P_J/U_{J(<l')}$.

For $\tau \in \mathrm{Irr}(V(l, l'))$ of rank r where $r \in J$ and $\ker(\tau) = w$, where w is a complement to the r-dimensional subspace stabilized by P^{+l}_r, there is a subsequent $e' = (J, \{l, l + r\}, (l, l'))$ with $\partial C' = (l, r)$. Then by definition

$$\begin{aligned} P(e') \cong& L^{n-2l'}_{J_0} \times P^{+(l-r)}_{J_1(e')} \times P^{+(l'-l)}_{J(\leq r)} \\ \cong& L^{n-2l'}_{J_0} \times P^{+(l-r)}_{J_1(e')} \times \left(P^{+r}_{J(<r)} \times \mathrm{GL}_{l'-(l+r)}(q^2)\right) \ltimes V(e') \end{aligned}$$

We have

$$T_{P(e)}(\tau) \cong P(e') \ltimes V(e).$$

As a block subgroup embedded in $P(e)/V(e)$, the $P^{+r}_{J(<r)}$ term in the last factor $P^{+(l'-l)}_{J(\leq r)}$ of $P(e')$ occurs twice. Thus the map $\phi_{e'}$ is the determinant on $P(e')$ as a subgroup in P_J.

If $l + r < l'$, then $P^{+(l'-l)}_{J(\leq r)}$ contains a nontrivial submodule isomorphic to $V(e') \cong M_{r,l'-(l+r)}(q^2)$. Suppose $J(\leq r)$ is nonempty. Then there exists $s \in J(\leq r)$ and $\tau' \in \mathrm{Irr}(V(e'))$ of rank s and $\ker(\tau')$ is a complement w' to the s-dimensional subspace stabilized by P^{+r}_s. There is a further triple $e'' = (J, \{l, l + r, l + r + s\}, (l, l'))$ with $\partial C'' = (l, r, s)$. By definition

$$\begin{aligned} P(e'') \cong& L^{n-2l'}_{J_0} \times P^{+(l-r)}_{J_1} \times P^{+(r-s)}_{J_2} \times P^{+(l'-(l+r))}_{J(\leq s)} \\ \cong& L^{n-2l'}_{J_0} \times P^{+(l-r)}_{J_1} \times P^{+(r-s)}_{J_2} \times \left(P^{+s}_{J(<s)} \times \mathrm{GL}_{l'-(l+r+s)}(q^2)\right) \ltimes V(e''). \end{aligned}$$

We have

$$T_{P(e')}(\tau') \cong P(e'') \ltimes V(e').$$

As a block subgroup embedded in $P(e)/V(e)$, the $P^{+s}_{J(<s)}$ term in the last factor $P^{+(l'-l)}_{J(\leq r)}$ of $P(e'')$ occurs three times. Thus the map $\phi_{e''}$ is the determinant on $P(e'')$ as a subgroup in P_J.

Remark: The collection of all $e \in E$ with fixed J and (l, l') unravels the alternating sum involving characters $\chi \in \mathrm{Irr}(P_J)$ that correspond to characters in $\mathrm{Irr}(V(l, l'))$. Informally, we describe the method as taking stabilizers in stabilizers in stabilizers, etc. At each step we mod out the involved interior (general linear) module until the only interior (general linear) module is trivial. This occurs when the last element of C is l'. Of course this description misses all the important details. The length of C keeps track of how many times we iterate this process. The partition ∂C keeps track of the rank of the characters for which we calculate stabilizers.

3.2 The elements of F and their related objects

Let f be an ordered triple (J, C, l) where either $f = (\emptyset, \emptyset, 0)$ or J, C, and l satisfy the following conditions:

1. The sequence $J = \{j_1, j_2, \ldots, j_r\}$ is a subset of $I = [m]$. We will assume that J is enumerated in increasing order.
2. The sequence $C = \{l_1, l_2, \ldots, l_s\}$ is a subset of $I = [m]$ also enumerated in increasing order. The sequence C must satisfy the same conditions as listed in the previous section. So C must be convex and the members of the related sequence ∂C appear in J.
3. The integer $l = j_r$ and $l = l_1$, so that l is the maximal member of J and the minimal member of C.

Notice that J and C are defined as they were for the triples $e \in E$. Let F be the set of all such f. The length of f is $l(f) = |C| = s$ and the parity of f is $|f| = |J|$ just as we did for $e \in E$.

Write $\partial C = (\partial l_1, \partial l_2, \ldots, \partial l_s)$ as in definition 3.2. Notice that the ∂l_i's are contained in J with $l = \partial l_1 = l_1$. Moreover l is the largest member of J. Thus the sequence $\partial l_s \leq \partial l_{s-1} \leq \cdots \leq \partial l_2 \leq \partial l_1 = l$ divides J into a collection of s disjoint subsets

$$\{j \in J | j < \partial l_s\}, \quad \cdots, \quad \{j \in J | \partial l_2 < j < \partial l_1 = l\}.$$

Definition 3.5 *We define*

1. $J_s = \{j \mid j \in J$ *and* $j < \partial l_s\}$*, and*
2. $J_i = \{j - \partial l_{i+1} \mid j \in J$ *and* $\partial l_{i+1} < j < \partial l_i\}$ *for* $1 \leq i < s$.

Notice that we have the containment $J_s \subset [\partial l_s - 1]$ and $J_i \subset [\partial l_i - \partial l_{i+1} - 1]$ for $1 \leq i < s$. We are now ready to define the groups related to the triple $f = (J, C, l)$.

Definition 3.6 *For $f \in F$ we define the following objects.*

1. *Let*

$$P(f) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times P^{n_s(f)}_{J_s(f)}$$

where

   (a) $n_i(f) = \partial l_i - \partial l_{i+1}$ *and* $J_i(f) = J_i$ *for* $1 \leq i < s$

   (b) $n_s(f) = n - 2l_{s-1}$ *and* $J_s(f) = J_s \cup \{\partial l_s\} = J(\leq \partial l_s)$

Notice that the last factor is a parabolic subgroup of a unitary group, whereas the first $s-1$ *factors are parabolic subgroups of general linear groups.*

2. *We define a normal subgroup of* $P(f)$*. The group* $P^{n_s(f)}_{J_s(f)}$ *is contained in the maximal parabolic* $P^{n_s(f)}_{\partial l_s}$ *which has unipotent radical* $U^{n_s(f)}_{\partial l_s}$*, a normal subgroup of* $P^{n_s(f)}_{J_s(f)}$ *and hence of* $P(f)$*. We set* $U(f) = U^{n_s(f)}_{\partial l_s}$*. Let* $Z(f) = Z(U(f))$ *and the quotient* $V(f) = U(f)/Z(f)$*. Observe that*

$$P^{n_s(f)}_{J_s(f)} \cong \left(P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l_s}(q)\right) \ltimes U(f) \tag{12}$$

Moreover $V(f)$ *is a unitary linear module for* $P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l_s}(q)$ *as discussed in Section 2.5 and* $Z(f)$ *is a central module for* $P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l_s}(q)$ *as discussed in Section 2.7.*

3. *We define*

$$d(f) = 2\sum_{i=1}^{s-1}\left(\binom{\partial l_i}{2} - \binom{\partial l_i - l_{i+1}}{2}\right).$$

Notice that $d(f)$ *depends only on the sequence* C*.*

4. *Finally we define a map* $\phi_f : P(f) \to F_{q^2}$*. For* $g \in P(f)$*, write* $g = A_1 \cdots A_{s-1}A_s$ *where* $A_i \in P^{+n_i(f)}_{J_i(f)}$ *for* $1 \leq i < s$ *and* $A_s \in P^{n_s(f)}_{J_s(f)}$*. We have a decomposition of* A_s*. Write* $A_s \equiv A_{s,1}A_{s,2}\mathrm{mod}(U(f))$ *where* $A_{s,1} \in P^{+\partial l_s}_{J_s}$ *and* $A_{s,2} \in \mathrm{U}_{n-2l}(q)$*. Let*

$$\phi_f(g) = \left[\left(\prod_{i=1}^{s-1}(\det A_i)^i\right)(\det A_{s,1})^s\right]^{1-q} \det A_{s,2} \tag{13}$$

where det *denotes the usual determinant map. Notice* $\phi_f(P(f)) \leq \mathbb{C}_{q+1}$*.*

In the context of unitary modules (both linear and central) the map $\mathcal{D}$ *defined in Sections 2.5 and 2.7 on* $P^{n_s(f)}_{J_s(f)}$ *corresponds to* $i = s$ *and is the restriction of* ϕ_f *to* $P^{n_s(f)}_{J_s(f)}$*.*

An example of F: Fix J with maximal member l. Set the initial $f = (J, \{l\}, l)$. Then $\partial C = (l)$ and

$$\begin{aligned} P(f) \cong P_J &\cong (P_J/U_l) \ltimes U_l \\ &\cong \left(P^{+l}_{J(<l)} \times \mathrm{U}_{n-2l}(q)\right) \ltimes U(f) \end{aligned}$$

where $U(f) \cong U_l$, $V(f) \cong U_l/Z_l$, $Z(f) \cong Z_l$, and under this isomorphism ϕ_f is the determinant map on P_J.

If $n \neq 2l$ then U_l/Z_l is not trivial. For $\tau \in \mathrm{Irr}(U_l/Z_l) \cong \mathrm{Irr}(V(f))$ of rank r and $\ker(\tau) = w$, where w is a complement to the r-dimensional space stabilized by P_r^{+l}, corresponding to a singular chain of type $J(< r)$ where $r \in J$, there is a subsequent $f' = (J, \{l, l+r\}, l)$ with $\partial C' = (l, r)$. Then by definition

$$\begin{aligned} P(f') \cong & P_{J_1(f')}^{+(l-r)} \times P_{J(\leq r)}^{n-2l} \\ \cong & P_{J_1(f')}^{+(l-r)} \times \left(P_{J(<r)}^{+r} \times \mathrm{U}_{n-2(l+r)}(q) \right) \ltimes U(f'). \end{aligned}$$

We have

$$T_{P(f)}(\tau) \cong P(f') \ltimes U(f).$$

As a block subgroup embedded in $P(f)/U(f)$, the $P_{J(<r)}^{+r}$ term in the last factor $P_{J(\leq r)}^{n-2l}$ of $P(f')$ occurs twice. Thus the map $\phi_{f'}$ is the determinant on $P(f')$ as a subgroup in P_J.

If $2(l+r) < n$, then $P_{J(\leq r)}^{n-2l}$ contains a submodule isomorphic to $U(f')$. If $V(f')$ is not trivial and $\mathrm{Irr}(V(f'))$ contains a character τ' with $\ker(\tau') = w'$, where w' is a complement to the s-dimensional space stabilized by P_s^{+r}, and τ' corresponds to a singular chain of rank s and type $J(< s)$ then if $s \in J$ we have $f'' = (J, \{l, l+r, l+r+s\}, l)$ with $\partial C'' = (l, r, s)$. Checking with the definition for f'' we have

$$\begin{aligned} P(f'') \cong & P_{J_1}^{+(l-r)} \times P_{J_2}^{+(r-s)} \times P_{J(\leq s)}^{n-2(l+r)} \\ \cong & P_{J_1}^{+(l-r)} \times P_{J_2}^{+(r-s)} \times \left(P_{J(<s)}^{+s} \times \mathrm{U}_{n-2(l+r+s)}(q) \right) \ltimes U(f'') \end{aligned}$$

We have

$$T_{P(f')}(\tau') \cong P(f'') \ltimes U(f').$$

As a block subgroup embedded in $P(f)/V(f)$, the $P_{J(<s)}^{+s}$ term in the last factor $P_{J(\leq s)}^{n-2(l+r)}$ of $P(f'')$ occurs three times. Thus the map $\phi_{f''}$ is the determinant on $P(f'')$ as a subgroup in P_J.

Remark: The collection of all $f \in F$ with fixed J with maximal element l unravels the alternating sum involving characters $\chi \in \mathrm{Irr}(P_J)$ that correspond to characters in $\mathrm{Irr}(U_l)$. Informally, we describe the method as taking stabilizers of singular flags in stabilizers of singular flags in stabilizers of singular flags, etc. At each step we mod out the involved interior (unitary linear) module until the remaining interior (unitary linear) module is trivial. This occurs when the last element of C is n. Of course this description misses all the important details. The length of C keeps track of how many times we iterate this process. The partition ∂C keeps track of the rank of the characters corresponding to singular flags for which we calculate stabilizers.

3.3 Results concerning members of E and F

There are a number of results that we need regarding members of E and F. These results lead to the unraveling of our alternating sum which in turn leads to some very nice cancellation. These are primarily modifications of Ku's results ([15], Chapter 8); however we have the added parameters det and j. To that end in this section we will define integers j_e and j_f which codify the splitting of characters upon restriction to the kernel of the determinant map. Regarding convex chains C and C', if $C = C' \backslash \{\max C'\}$ we say that C' covers C and write $C \prec C'$.

We begin with members of the set E. Let C be a sequence with $|C| = s$ and $l_s < l'$. Fix $e = (J, C, (l, l'))$ and $e' = (J, C \cup \{l_s + r\}, (l, l'))$ in E, where $1 \le r \le \min\{l_s, l' - l_s\}$, $r \in J$, and $l_s + r \le l'$. Then $C \prec C \cup \{l_s + r\}$. Notice $l(e) = s$ and $l(e') = s + 1$. Take $\tau \in \mathrm{Irr}(V(e), r)$ where $\ker(\tau)$ is a complement w to the r-dimensional space stabilized by $P_r^{+\partial l_s}$. Then

$$T_{P(e)}(\tau) = L_{J_0(e)}^{n_0(e)} \times P_{J_1(e)}^{+n_1(e)} \times \cdots \times P_{J_{s-1}(e)}^{+n_{s-1}(e)} \times T_{P_{J_s(e)}^{+n_s(e)}}(\tau)$$

and $T_{P_{J_s(e)}^{+n_s(e)}}(\tau) \cong \left(P_{J_s(e')}^{+n_s(e')} \times P_{J_{s+1}(e')}^{+n_{s+1}(e')} \right) \ltimes V(e)$. Thus

$$T_{P(e)}(\tau) \cong P(e') \ltimes V(e)$$

and ϕ_e restricted to $T_{P(e)}(\tau)$ is $\phi_{e'}$. Hence

$$k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j) = k_{d-d(e')}(P(e'), \rho, \phi_{e'}, j') \tag{14}$$

where j' the least positive integer such that j divides $j' \cdot |T_{P(e)}(\tau) \ker(\phi_e) \backslash P(e)|$. Write $T = T_{P(e)}(\tau)$, $K = \ker(\phi_e)$. Then $K \trianglelefteq P(e)$ so

$$TK \big/ K \cong T \big/ T \cap K \cong T/V(e) \Big/ (T \cap K)/V(e) \cong P(e') \Big/ \ker(\phi_{e'})$$

Moreover $P(e)/K \cong \phi_e(P(e))$ and $P(e')/\ker(\phi_{e'}) \cong \phi_{e'}(P(e'))$ and hence

$$|TK \backslash P(e)| = \left| P(e) \big/ TK \right| = \left| P(e)/K \Big/ TK/K \right| = \left| \phi_e(P(e)) \Big/ \phi_{e'}(P(e')) \right|$$

Since $l_s < l'$, i.e. $l' - l_s \neq 0$ the image of $P(e)$ under ϕ_e is $\mathcal{C}_{q+1}$. Thus we have

$$|TK \backslash P(e)| = \frac{|\phi_e(P(e))|}{|\phi_{e'}(P(e'))|} = \frac{q+1}{|\phi_{e'}(P(e'))|}.$$

Definition 3.7 *Let j_e be the smallest positive integer such that j divides $j_e \cdot \dfrac{q+1}{|\phi_e(P(e))|}$.*

Remark: Observe that for e of length 1, $j_e = j$ certainly holds. In general, for fixed $e \in E$, $|\phi_e(P(e))|$ is almost always equal to $q+1$ and hence $j_e = j$. The image

$$\phi_e(P(e)) = \prod_{i=0}^{s}(\mathbb{C}_{h_i})$$

where $\mathbb{C}_{h_i}$ is the image of the i-the factor and so depends on the $n_i(e)$. If any of $n_0(e)$, $n_1(e)$, or $l' - l_s$ are nonzero, then $|\phi_e(P(e))| = q+1$. If all three are zero then C cannot be covered and

1. $n = 2m$, $l' = m$, and $l_s = m$
2. $C = \{l, 2l, \ldots, m\}$ so that Ç begins $(l, l, \ldots)$.

On the other hand if C can be covered, and $e = (J, C, (l, l'))$ and $e' = (J, C \cup \{l_s + r\}, (l, l'))$ are members of E as in 14, then

$$k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j) = k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j_e) = k_{d-d(e')}(P(e'), \rho, \phi_{e'}, j_{e'}).$$

Finally, for fixed e with $l(e) \geq 2$

$$k_{d-d(e)}(P(e), \rho, \phi_e, j_e) = k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) + k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e)$$

where $k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) = 0$ if $V(e) = 1$ i.e. $l_s = l'$.

By the first reduction in Proposition 2.6, when summing over all e of length 1 and $J \subseteq I$ of the form $J = J' \cup J''$ where $J' \subseteq [l-1]$ varies and $J'' \subset I$ is fixed with minimal member l', we need only sum over $\tau \in \mathrm{Irr}(V(e), r)$ with $r \in J'$ and $\ker(\tau) = w$, where w is a complement to the space stabilized by P_r^{+l}. By repeated application of this reasoning it follows that

Proposition 3.8

$$\sum_{\substack{e \in E \\ l(e)=1}} (-1)^{|e|} k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j) = \sum_{\substack{e \in E \\ l(e)\geq 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e).$$

Now we turn to results concerning members of F. Let C be a sequence with $|C| = s$ and $l_s < m$. Fix $f = (J, C, l)$ and $f' = (J, C \cup \{l_s + r\}, l)$ in F, where $1 \leq r \leq \min\{l_s, n - 2l_s\}$, $r \in J$, and $l_s + r \leq m$. Then $C \prec C \cup \{l_s + r\}$. Notice $l(f) = s$ and $l(f') = s + 1$. Take $\tau \in \mathrm{Irr}(V(f))$ corresponding to a singular chain of rank r. Then

$$T_{P(f)}(\tau) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times T_{P^{n_s(f)}_{J_s(f)}}(\tau)$$

and $T_{P^{n_s(f)}_{J_s(f)}}(\tau) \cong \left(P^{+n_s(f')}_{J_s(f')} \times P^{n_{s+1}(f')}_{J_{s+1}(f')}\right) \ltimes U(f)$. Thus

$$T_{P(f)}(\tau) \cong P(f') \ltimes U(f)$$

and ϕ_f restricted to $T_{P(f)}(\tau)$ is $\phi_{f'}$. Hence

$$k_{d-d(f)}(P(f),\tau,\rho,\phi_f,j) = k_{d-d(f')}(P(f'),\rho,\phi_{f'},j') \tag{15}$$

where j' the least positive integer such that j divides $j' \cdot |T_{P(f)}(\tau)\ker(\phi_f)\backslash P(f)|$. Write $T = T_{P(f)}(\tau)$, $K = \ker(\phi_f)$. Then $K \trianglelefteq P(f)$ so

$$TK\,/K \cong T\,/T\cap K \cong T/U(f)\Big/(T\cap K)/U(f) \cong P(f')\Big/\ker(\phi_{f'})$$

Moreover $P(f)/K \cong \phi_f(P(f))$ and $P(f')/\ker(\phi_{f'}) \cong \phi_{f'}(P(f'))$ and hence

$$|TK\backslash P(f)| = \left|P(f)\,/TK\right| = \left|P(f)/K\Big/TK/K\right| = \left|\phi_f(P(f))\Big/\phi_{f'}(P(f'))\right|$$

Since $l_s < m$ we have nonzero $n - 2l_s$, thus the image of $P(f)$ under ϕ_f is $\mathcal{C}_{q+1}$ so we have

$$|T_{P(f)}(\tau)\ker(\phi_f)\backslash P(f)| = \frac{|\phi_f(P(f))|}{|\phi_{f'}(P(f'))|} = \frac{q+1}{|\phi_{f'}(P(f'))|}.$$

Definition 3.9 *Let j_f be the smallest positive integer such that j divides $j_f \cdot \frac{q+1}{|\phi_f(P(f))|}$.*

Remark: Observe that for f of length 1, $j_e = j$ certainly holds. In general, for fixed $f \in F$, $|\phi_f(P(f))|$ is almost always equal to $q+1$ and hence $j_f = j$. The image

$$\phi_f(P(f)) = \prod_{i=1}^{s}(\mathbb{C}_{h_i})$$

where $\mathbb{C}_{h_i}$ is the image of the i-the factor and so depends on $n_i(f)$. For $1 \leq i < s$, if $n_i(f) = 0$ then $h_i = 1$ whereas if $n_i(f) \neq 0$, $h_i = q+1/\gcd(q+1,i)$. In the s-th factor, if $n - 2l_s = 0$ then $h_s = q+1/\gcd(q+1,s)$ whereas if $n - 2l_s \neq 0$ then $h_s = q+1$.

$$|\phi_f(P(f))| = \mathbb{C}_L \text{ where } L = \mathrm{lcm}(h_i)$$

If either of $n_1(f)$ or $n - 2l_s$ are nonzero, then $|\phi_f(P(f))| = q+1$. If both are zero then C cannot be covered and

1. $n = 2m$ and $l_s = m$
2. $C = \{l, 2l, \ldots, m\}$ so that Ç begins $(l, l, \ldots)$.

On the other hand, if C can be covered and $f = (J, C, l)$ and $f' = (J, C \cup \{l_s + r\}, l)$ are in F as in 15 then

$$k_{d-d(f)}(P(f),\tau,\rho,\phi_f,j) = k_{d-d(f)}(P(f),\tau,\rho,\phi_f,j_f) = k_{d-d(f')}(P(f'),\rho,\phi_{f'},j_{f'}).$$

We are interested in counting irreducible characters of $P(f)$ that correspond to irreducible characters of $U(f)$. The following is analogous to definitions 2.10, 2.12, and 2.19 in the discussion of unitary linear modules and central modules in the previous section.

Definition 3.10 *1. Let $S^u(f)$ denote the subset of nonidentity characters in* $\mathrm{Irr}(U(f))$ *which are trivial on $Z(f)$.*

2. Let $S^{su}(f)$ denote the subset of characters in $\mathrm{Irr}(U(f))$ *which are trivial on $Z(f)$ and correspond to singular flags in the unitary space V_2.*

3. Let $S^{nu}(f)$ denote the subset of characters in $\mathrm{Irr}(U(f))$ *which are trivial on $Z(f)$ and correspond to nonsingular flags in the unitary space V_2.*

4. Let $S^z(f)$ denote the subset of characters in $\mathrm{Irr}(U(f))$ *which are not trivial on $Z(f)$.*

Clearly $\mathrm{Irr}(U(f)) = 1 \cup S^u(f) \cup S^z(f) = 1 \cup S^{su}(f) \cup S^{nu}(f) \cup S^z(f)$ holds.

For fixed f with $l(f) \geq 2$

$$k_{d-d(f)}(P(f), \rho, \phi_f, j_f) = k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) + k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f)$$

and

$$\begin{aligned} k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) =& k_{d-d(f)}(P(f), S^{su}(f), \rho, \phi_f, j_f) \\ &+ k_{d-d(f)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) \\ &+ k_{d-d(f)}(P(f), S^z(f), \rho, \phi_f, j_f). \end{aligned}$$

Observe that if X is the subset of $S^{su}(f)$ containing only characters of rank r, where $r \in J$, and kernels equal to complements of spaces stabilized by $P_r^{+\partial l_s}$, then

$$k_{d-d(f)}(P(f), X, \rho, \phi_f, j_f) = \sum_{\substack{f'=(J,C',l) \\ C \prec C'}} k_{d-d(f')}(P(f'), \rho, \phi_{f'}, j_{f'}).$$

By the first reduction in Proposition 2.6, when summing over all f of length 1 and $J \subseteq I$ has maximal element l, we need only sum over $\tau \in S^{su}(f)$ with $r \in J$ and $\ker(\tau) = w$, where w is a complement to the space stabilized by P_r^{+l}. By repeated application of this reasoning it follows that

Proposition 3.11

$$\begin{aligned} \sum_{\substack{f \in F \\ l(f)=1}} (-1)^{|e|} k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j) = \\ \sum_{\substack{f \in F \\ l(f)\geq 2}} (-1)^{|f|} k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) + \\ \sum_{\substack{f \in F \\ l(f)\geq 1}} (-1)^{|e|} k_{d-d(e)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) + \\ \sum_{\substack{f \in F \\ l(f)\geq 1}} (-1)^{|e|} k_{d-d(e)}(P(f), S^z(f), \rho, \phi_f, j_f). \end{aligned} \tag{16}$$

Our next result involves the members of E and F. This is a modification of Ku's result. We have the additional parameters ϕ_e, ϕ_f and j_e, j_f. This remarkable cancellation leads to the second reduction in 1b .

Proposition 3.12

$$\sum_{\substack{e\in E\\ l(e)\geq 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) + \sum_{\substack{f\in F\\ l(f)\geq 2}} (-1)^{|f|} k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) = 0$$

Proof: We proceed by explicitly matching pairs in the sum with opposite parity. There are four cases.

1. Match $f = (J, C, l)$ to $e = (J \cup \{l_s\}, C, (l, l_s))$. This is possible since $s \geq 2$.

$$P(f) \cong P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times \left(P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l_s}(q)\right) \ltimes U(f)$$

   Then $n_0(e) = n - 2l_s$ and $J_0(e) = \emptyset$, $n_s(e) = l_s - l_{s-1} = \underset{\cdot}{l}_s$ so $V(e) = 1$. Moreover $n_i(f) = n_i(e)$ for $1 \leq i < s$ and we have

$$P(e) \cong \mathrm{U}_{n-2l_s}(q) \times P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times \left(P^{+\partial l_s}_{J_s} \times 1\right) \ltimes 1.$$

   Thus

$$P(f)/U(f) \cong P(e)/V(e) \quad \text{and} \quad |\phi_f(P(f))| = |\phi_e(P(e))|$$

   since ϕ_f and ϕ_e agree on isomorphic factors of the quotient groups.

2. Match $e = (J, C, (l, l'))$ to $e' = (J \cup \{l_s\}, C, (l, l_s))$, if $l_s < l'$.

$$P(e) \cong \mathrm{L}^{n-2l'}_{J_0} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times \mathrm{GL}_{l'-l_s}(q^2)\right) \ltimes V(e).$$

$$P(e') \cong \mathrm{L}^{n-2l_s}_{J_0} \times P^{+n_1(e')}_{J_1(e')} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times 1\right) \ltimes 1.$$

   Moreover since there are no elements in $J \cup \{l_s\}$ between l_s and l'

$$L^{n-2l_s}_{J_0(e')} \cong L^{n-2l'}_{J_0(e)} \times \mathrm{GL}_{l'-l_s}(q^2).$$

   Thus

$$P(e)/V(e) \cong P(e')/V(e') \quad \text{and} \quad |\phi_e(P(e))| = |\phi_{e'}(P(e'))| = q + 1.$$

3. Match $e = (J, C, (l, l'))$ to $e' = (J \backslash \{l'\}, C, (l, l''))$, if $l_s = l' < \max J$, where l, l', l'' are consecutive elements in J.

$$P(e) \cong \mathrm{L}^{n-2l'}_{J_0(e)} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times 1\right) \ltimes 1.$$

$$P(e') \cong Ł^{n-2l''}_{J_0(e')} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times \mathrm{GL}_{l''-l_s}(q^2)\right) \ltimes V(e').$$

Moreover since there are no elements in $J \backslash \{l'\}$ between $l_s = l'$ and l''

$$L^{n-2l'}_{J_0(e)} \cong L^{n-2l''}_{J_0(e')} \times \mathrm{GL}_{l''-l_s}(q^2).$$

and $n - 2l' \neq 0$. Thus

$$P(e)/V(e) \cong P(e')/V(e') \quad \text{and} \quad |\phi_e(P(e))| = |\phi_{e'}(P(e'))| = q + 1.$$

4. Match $e = (J, C, (l, l'))$ to $f = (J \backslash \{l'\}, C, l)$, if $l_s = l' = \max J$.

$$P(e) \cong Ł^{n-2l'}_{J_0(e)} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times 1\right) \ltimes 1$$

where $J_0(e) = \emptyset$. Then $n_i(e) = n_i(f)$ for $1 \leq i < s$

$$P(f) \cong P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times \left(P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l'}(q)\right) \ltimes U(f)$$

Thus

$$P(e)/V(e) \cong P(f)/U(f) \quad \text{and} \quad |\phi_e(P(e))| = |\phi_f(P(f))|$$

since ϕ_e and ϕ_f agree on isomorphic factors of the quotient groups.

And we are done.

4 Completion of the Verification

Let ρ be an irreducible character of the center of $\mathrm{U}_n(q)$. Recall $k_d^1(P_J, U_J, \rho, \det, j)$ is the number of irreducible characters $\chi \in \mathrm{Irr}(P_J)$ such that the unipotent radical U_J is not contained in $\ker \chi$, χ lies over ρ, and χ restricted to $\ker \det$ is a sum of j' irreducible characters where j divides j'.

In this section we prove 1b which we restate here

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) = - \sum_{\substack{\mu \vdash n \\ n'(\mu) = d \\ j | \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho). \tag{17}$$

This completes the proof of DOC for the finite special unitary groups.

We begin by reformulating the left hand side of 17 via our parametrization using E and F from the previous section. After this reformulation we will refine the statement. Lastly we will need to make use of several layers of inductive arguments. To that end we introduce several propositions which are modifications of Ku's results. The extra parameters ϕ_f and j_f require even more involved combinatorial details.

4.1 The Reformulation

Recall, from the example on page 32 we have

$$k_d^1(P_J, V(j_i, j_{i+1}), \rho, \det, j) = k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e)$$

where $e = (J, \{j_i\}, (j_i, j_{i+1}))$ and from the example on page 34 we have

$$k_d^1(P_J, U_{j_s}, \rho, \det, j) = k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f)$$

where $f = (J, \{j_s\}, j_s)$. Hence for fixed J we have the decomposition

$$k_d^1(P_J, U_J, \rho, \det, j) = \sum_{\substack{e=(J,C,t) \\ l(e)=1}} k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j) + k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j). \tag{18}$$

Recall our definition of parity $|e|$ and $|f|$. Also note that for e and f of length 1, $d(e) = 0$, $d(f) = 0$, $j_e = j$, and $j_f = j$. Thus the left hand side of 17 can be written

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) =$$

$$\sum_{\substack{e \in E \\ l(e)=1}} (-1)^{|e|} k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j) + \sum_{\substack{f \in F \\ l(f)=1}} (-1)^{|f|} k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j) \tag{19}$$

We have the further reduction by Proposition 3.8

$$\sum_{\substack{e\in E\\ l(e)=1}} (-1)^{|e|} k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j) = \sum_{\substack{e\in E\\ l(e)\geq 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e).$$

Moreover by Proposition 3.11,

$$\begin{aligned}\sum_{\substack{f\in F\\ l(f)=1}} (-1)^{|e|} k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j) = \\ \sum_{\substack{f\in F\\ l(f)\geq 2}} (-1)^{|f|} k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) + \\ \sum_{\substack{f\in F\\ l(f)\geq 1}} (-1)^{|e|} k_{d-d(e)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) + \\ \sum_{\substack{f\in F\\ l(f)\geq 1}} (-1)^{|e|} k_{d-d(e)}(P(f), S^{z}(f), \rho, \phi_f, j_f).\end{aligned} \tag{20}$$

Recall the cancellation of Proposition 3.12, i.e.

$$\sum_{\substack{e\in E\\ l(e)\geq 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) + \sum_{\substack{f\in F\\ l(f)\geq 2}} (-1)^{|f|} k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) = 0$$

Thus, we can now omit all the terms in our sum where the characters correspond either to characters of so called general linear modules, or to the characters of so called unitary linear modules which themselves correspond to singular flags, i.e. all the characters of the $P(e)$'s together with the characters of the $P(f)$'s which correspond to characters of $V(f)$ corresponding to singular flags in a unitary vector space. We are left with characters of the $P(f)$'s that correspond to flags of nonsingular type for $V(f)$ and characters of $Z(f)$. The left hand side of 17 may thus be written

$$\begin{aligned}\sum_{J\subseteq I} (-1)^{|J|} k^1_d(P_J, U_J, \rho, \det, j) = \\ \sum_{\substack{f\in F\\ l(f)\geq 1}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) + \\ \sum_{\substack{f\in F\\ l(f)\geq 1}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{z}(f), \rho, \phi_f, j_f).\end{aligned} \tag{21}$$

4.2 The refinement

At this stage in order to proceed in our calculations we must separate the set of $P(f)$'s according to the length of f. Recall that $l(f) = |C|$ which keeps track of the number of times we have iterated the process of taking stabilizers. In this sense C also keeps track of the structure of $P(f)$ as a product of block subgroups embedded in $\mathrm{U}_n(q)$. Recalling definition 3.5 of $P(f)$, C keeps track of the number of times each component of $P(f)$ appears in $\mathrm{U}_n(q)$. By collecting all the f of the same length s, we are gathering all the $P(f)$ which have the same structure as products of block subgroups embedded in $\mathrm{U}_n(q)$.

Proposition 4.1 *For fixed n and s, $1 \leq s \leq m$ the following hold:*

$$\sum_{\substack{f\in F\\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) = - \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)=2s+1\\ j|\gcd(\lambda(\mu),q+1)}} \beta(\mu, a_\rho) \tag{22}$$

$$\sum_{\substack{f\in F\\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{z}(f), \rho, \phi_f, j_f) = - \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)=2s\\ j|\gcd(\lambda(\mu),q+1)}} \beta(\mu, a_\rho) \tag{23}$$

Summing over all s, Proposition 4.1 certainly implies 17. However, this is not enough. In order to prove this proposition we must further decompose the sets $S^{nu}(f)$ and $S^z(f)$. As has been discussed, originally in section 2 and then in section 3, characters in $S^{nu}(f)$ have non-singular rank by definition. Characters in $S^z(f)$ have non-singular rank if they do not correspond to singular flags. For clarity observe that in section 2 the roles of r and r' were reversed.

Recall ∂C kept track of the rank of singular unitary characters for which we took stabilizers. At each iteration of the process the rank of the next singular character could not exceed the rank of the previous character. In fact ∂C is precisely the list of these ranks in order reading from left to right. We are done calculating stabilizers of singular unitary characters. However, we are not done calculating stabilizers for $\tau \in S^{nu}(f)$ or $S^z(f)$. For fixed r, if the minimal element of ∂C is greater than or equal to r, then the last factor of $P(f)$ is

$$P^{n_s(f)}_{J_s(f)} \cong \left(P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l_s}(q)\right) \ltimes U(f)$$

where $\partial l_s \geq r$. Hence $P(f)$ is big enough to contain $\tau \in S^{nu}(f)$ or $S^z(f)$ of non-singular rank r.

Now this is the key fact: Given τ of non-singular rank $r \in J$, the stabilizer of τ in the last factor of $P(f)$ contains a subgroup isomorphic to $P^r_{J(<r)}$. By gathering together the f of the same length with $\partial C \geq r$ we are grouping all the $P(f)$ with the same block structure as embedded in $\mathrm{U}_n(q)$ which have stabilizers containing copies of $P^r_{J(<r)}$. Hence we will be able to peel off this factor and make use of a layered inductive argument.

Recall from definition 3.10, $S^{nu}(f)$ is the set of irreducible characters of the unitary linear module $V(f)$ that correspond to nonsingular flags.

Definition 4.2 *We define the following subsets.*

1. *Let $S_r^{nu}(f)$ denote the elements of $S^{nu}(f)$ with non-singular rank r.*

2. *Let $S_r^{nu}(f)(r')$ denote the elements of $S_r^{nu}(f)$ with rank r'.*

In keeping with the original notation of definition 2.12 from section 2, we have $S_r^{nu}(f)(r') = S_r^{nu}(V(f), J(< \partial l_s), r')$. Notice that

$$S_r^{nu}(f) = \bigcup_{r'=r}^{\min(\partial l_s, n-2l_s)} S_r^{nu}(f)(r').$$

Recall from definition 3.10, $S^z(f)$ is the set of nontrivial irreducible characters of the central module $Z(f)$.

Definition 4.3 *We define the following subsets.*

1. *let $S_r^z(f)$ denote the elements of $S^z(f)$ with non-singular rank r.*

2. *Let $S_r^z(f)(r')$ denote the elements of $S_r^z(f)$ with rank r'.*

In keeping with the original notation of definition 2.21 from section 2, we have $S_r^z(f)(r') = S_r^z(Z(f), J(< \partial l_s), r')$. Also recall that from this definition $S_r^z(f)(r)$ is the set of elements of $S^z(f)$ labeled by singular chains. Notice that

$$S_r^z(f) = \bigcup_{r'=r}^{\partial l_s} S_r^z(f)(r').$$

Summing over all r, the following proposition certainly implies Proposition 4.1.

Proposition 4.4 *For fixed n, s, and r with $1 \leq s, r \leq m$ the following hold:*

$$\sum_{\substack{f \in F \\ l(f)=s \\ \min \partial C \geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) = - \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)=2s+1 \\ \min(\mu)=r \\ j | \gcd(\lambda(\mu), q+1)}} \beta(\mu, a_\rho) \tag{24}$$

$$\sum_{\substack{f \in F \\ l(f)=s \\ \min \partial C \geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S_r^{z}(f), \rho, \phi_f, j_f) = - \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)=2s \\ \min(\mu)=r \\ j | \gcd(\lambda(\mu), q+1)}} \beta(\mu, a_\rho) \tag{25}$$

However, at this point the above proposition seems somewhat convoluted. In general inductive arguments can be less than transparent. Thus, before continuing we briefly remark on why odd partitions are involved in 24 and even partitions are involved in 25 in the above proposition.

Remark: Take $\tau \in S_r^{nu}(f)$. Then $T_{P(f)}(\tau)$ contains an isomorphic copy of $P^r_{J(<r)}$ and the multiplicity of this factor as a block subgroup is $2s+1$. On the other hand, for $\tau \in S_r^z(f)$ $T_{P(f)}(\tau)$ contains an isomorphic copy of $P^r_{J(<r)}$ and the multiplicity of this factor as a block subgroup is $2s$. At bottom, when for example the set $J(<r)$ is empty so that $P^r_{J(<r)} = \mathrm{U}_r(q)$, we know from the remarks following definition 3.6 in [1] that irreducible characters of $\mathrm{U}_r(q)$ fall into classes of certain type given by partitions of r. These partitions encode the degree of the characters and also the splitting upon restriction to $\mathrm{SU}_r(q)$. Since $2s+1$ or $2s$ copies of $\mathrm{U}_r(q)$ (according to whether $\tau \in S_r^{nu}(f)$ or $\tau \in S_r^z(f)$) appear in $\mathrm{U}_n(q)$ it is not unreasonable to suppose that partitions of r will lead to partitions of n. For fixed C and l the left hand sums in 24 and 25 involve all f with $J = J' \cup J''$ where $J' \subseteq [r-1]$ varies and J'' with minimal member r is fixed. We will eventually be able to apply induction to the following sub-sums involved in 24 and 25

$$\sum_{J \subseteq [r-1]} (-1)^{|J|} k_{d-d(f)-d'}(P^r_J, \rho, \det{}^{2s+1}, j') \text{ in 24}$$

$$\sum_{J \subseteq [r-1]} (-1)^{|J|} k_{d-d(f)-d'}(P^r_J, \rho, \det{}^{2s}, j'') \text{ in 25.}$$

where $j' = \dfrac{j}{\gcd(j, q+1, 2s+1)}$ and $j'' = \dfrac{j}{\gcd(j, q+1, 2s)}$.

4.3 The rest

Before proving Proposition 4.4, we state and then prove an important corollary necessary for the inductive step. Moreover we present two intermediate propositions which allow us to rewrite the left hand sides of 24 and 25. We introduce two maps in order to keep track of the members of the involved alternating sums. These are extensions of maps defined by Ku ([15], Chapter 9). We use his notations: h, g.

In order to streamline the very involved notation we define certain subsets of F^n and then introduce two $\mathbb{Z}$-valued maps h, g on those subsets.

Definition 4.5 *For fixed n and s,*

1. *Let $F^n(s,r)$ denote the f in F^n with $l(f) = |C| = s$ and $\min \partial C \geq r$.*
2. *Let $F^n(\leq s)$ denote the $f \in F^n$ of length $l(f) \leq s$*

We now define our first map h which is involved in expressing the left hand side of 24

Definition 4.6 *Fix n and $1 \le r \le m$. Define $h_{n,d,\rho,s,r,j} : F^{n-(2s+1)r}(\le s) \to \mathbb{Z}$ by*

$$h_{n,d,\rho,s,r,j}(f) = \begin{cases} k^0_{d-d(f)}(P^{n-(2s+1)r}(f), U^{n-(2s+1)r}(f), \rho, \phi_f, j_f), & \text{if } 0 \le l(f) < s; \\ k^0_{d-d(f)}(P^{n-(2s+1)r}(f), Z^{n-(2s+1)r}(f), \rho, \phi_f, j_f), & \text{if } l(f) = s. \end{cases}$$

The following is a first crucial step which is involved in peeling off the factor $P^r_{J(<r)}$ in the left hand side of 24.

Proposition 4.7 *Fix n and $1 \le r \le m$. Then the alternating sum*

$$\sum_{\substack{f\in F \\ l(f)=s \\ \min\partial C \ge r}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{nu}_r(f), \rho, \phi_f, j_f) =$$

$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det{}^{2s+1}, j_1) \sum_{f\in F^{n-(2s+1)r}(\le s)} (-1)^{|f|} h_{n,d_2,\rho_2,s,r,j}(f)\right)$$

where $j_1 = j/\gcd(j, q+1, 2s+1)$, $d_1 + d_2 = d - d(f)$, and $\rho_1\rho_2 = \rho$.

Remark: If $n-(2s+1)r = 0$, then there is a unique $f = (\emptyset, \emptyset, 0) \in F^{n-(2s+1)r}(\le s)$. Moreover, we must set $j_f = 1$ regardless of the value of j in order that our sum may accommodate this degenerate case. Thus

$$\begin{aligned} h_{n,d,\rho,s,r,j}(f) &= k^0_{d-d(f)}(P^{n-(2s+1)r}(f), U^{n-(2s+1)r}(f), \rho, \phi_f, j_f) \\ &= k^0_d(1, 1, \rho, \det, 1) \\ &= 1 \text{ if and only if } \rho = 1 \text{ and } d = 0. \end{aligned}$$

Proof: Take $f \in F^n(s, r)$. We consider three cases.

1. Let $\partial C = \{r\}$. Then $C = (r, 2r, \ldots, sr)$ and

$$P(f) = \left(P^{+r}_{J(<r)} \times \mathrm{U}_{n-2sr}(g)\right) \ltimes U(f).$$

   $S^{nu}_r(f)$ is non-empty if and only if $n - 2sr \neq 0$ and $J(<r) \subset \{1, 2, \ldots, [\frac{n-2sr}{2}]\}$. If $S^{nu}_r(f) \neq \emptyset$ then it has one orbit. Take $\tau \in S^{nu}_r(f)$, then

$$T = T_{P(f)}(\tau) = \left(P^r_{J(<r)} \times \mathrm{U}_{n-2sr-r}(q)\right) \ltimes U(f).$$

   Thus $k_{d-d(f)}(P(f), S^{nu}_r(f), \rho, \phi_f, j_f) = k_{d-d(f)-d'}(T/U(f), \rho, \phi_f, j')$ where $j = j_f$, j' is the least positive integer such that j divides $j' \cdot \dfrac{q+1}{|\phi_f(T)|}$ and d' is q-height in the index of T in $P(f)$. We have $\mathrm{U}_{n-2sr-r}(q) = P^{n-(2s+1)r}(f')$ for $f' = (\emptyset, \emptyset, 0) \in F^{n-(2s+1)r}(\leq s)$ and

$$\begin{aligned} & k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) \\ & \qquad = \sum_{\rho_1,\rho_2} \sum_{d_1,d_2} k_{d_1}(P^r_{J(<r)}, \rho_1, (\det)^{2s+1}, j_1) k_{d_2}(P^{n-(2s+1)r}(f'), \rho_2, \phi_{f'}, j_{f'}) \\ & \qquad = \sum_{\rho_1,\rho_2} \sum_{d_1,d_2} k_{d_1}(P^r_{J(<r)}, \rho_1, (\det)^{2s+1}, j_1) h_{n,d_2,\rho_2,s,r,j}(f') \end{aligned}$$

where $j_1 = j/\gcd(j, q+1, 2s+1)$, $d_1 + d_2 = d - d(f)$, and $\rho_1\rho_2 = \rho$.

2. Let $\partial C \supsetneq \{r\}$. Then $\partial C = (l_1, \ldots, r)$ where $l_1 > r$.

$$P(f) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times P^{n_s(f)}_{J_s(f)}$$

where $n_i(f)$ is nonzero for some $1 \leq i < s$ and the last factor decomposes

$$P^{n_s(f)}_{J_s(f)} = \left(P^{+r}_{J(<r)} \times \mathrm{U}_{n-2l_s}(g)\right) \ltimes U(f).$$

$S_r^{nu}(f)$ is non-empty if and only if $n - 2l_s \neq 0$ and $J(<r) \subset \{1, 2, \ldots, [\frac{n-2l_s}{2}]\}$. If $S_r^{nu}(f) \neq \emptyset$ then it has one orbit. Take $\tau \in S_r^{nu}(f)$, then

$$T = T_{P(f)}(\tau) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times \left(P^r_{J(<r)} \times \mathrm{U}_{n-2l_s-r}(q)\right) \ltimes U(f).$$

Thus $k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) = k_{d-d(f)-d'}(T/U(f), \rho, \phi_f, j')$ where $j = j_f$, j' is the least positive integer such that j divides $j' \cdot \dfrac{q+1}{|\phi_f(T)|}$ and d' is q-height in the index of T in $P(f)$. Let $J' = \{j - r | j \in J(> r)\}$ and $C' = \{l_i - ir | l_i \in C\}$ $\{0\}$. For $f' = (J', C', l_1 - r) \in F^{n-(2s+1)r}(\leq s)$ we have

$$P^{n-(2s+1)r}(f') = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-2}(f)}_{J_{s-2}(f)} \times \left(P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times \mathrm{U}_{n-2l_s-r}(q)\right) \ltimes U^{n-(2s+1)r}(f').$$

Thus

$$\begin{aligned} & k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) \\ & \qquad = \sum_{\rho_1,\rho_2} \sum_{d_1,d_2} k_{d_1}(P^r_{J(<r)}, \rho_1, (\det)^{2s+1}, j_1) k^0_{d_2}(P^{n-(2s+1)r}(f'), U^{n-(2s+1)r}(f'), \rho_2, \phi_{f'}, j_{f'}) \\ & \qquad = \sum_{\rho_1,\rho_2} \sum_{d_1,d_2} k_{d_1}(P^r_{J(<r)}, \rho_1, (\det)^{2s+1}, j_1) h_{n,d_2,\rho_2,s,r,j}(f') \end{aligned}$$

where $j_1 = j/\gcd(j, q+1, 2s+1)$, $d_1 + d_2 = d - d(f)$, and $\rho_1\rho_2 = \rho$.

3. Let $\min(\partial C) > r$. Then $\partial l_s > r$. We have

$$P(f) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times P^{n_s(f)}_{J_s(f)}.$$

The last factor decomposes

$$P^{n_s(f)}_{J_s(f)} = \left(P^{+\partial l_s}_{J(<r)} \times \mathrm{U}_{n-2l_s}(g)\right) \ltimes U(f).$$

Assume $S^{nu}_r(f)$ is non-empty, so $n-2l_s \neq 0$ and $J(<r) \subset \{1,2,\ldots,[\frac{n-2l_s}{2}]\}$. Take $\tau \in S^{nu}_r(f)$ of rank r'. Then τ corresponds to a pair of chains (c_1,c_2) where c_1 is a singular chain in a unitary space of dimension r and c_2 is a flag in a unitary space of dimension $n-2l_s-r$. Let τ' correspond to c_2, then

$$T = T_{P(f)}(\tau) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times T_{P^{n_s(f)}_{J_s(f)}}(\tau)$$

where

$$T_{P^{n_s(f)}_{J_s(f)}}(\tau) = \left(P^{+(\partial l_s - r')}_{J_1} \times P^r_{J(<r)} \times T_{\mathrm{U}_{n-2l_s-r}(q)}(\tau')\right) \ltimes U(f).$$

Here $J_1 = \{j-r' | j \in J_1(f)(>r')\}$.

And we are done.

We now define our second map g which is involved in expressing the left hand side of 25.

Definition 4.8 *Fix n and $1 \leq r \leq m$. Define $g_{n,d,\rho,s,r,j} : F^{n-2sr}(\leq s) \to \mathbb{Z}$ by*

$$g_{n,d,\rho,s,r,j}(f) = \begin{cases} k^0_{d-d(f)}(P^{n-2sr}(f), U^{n-2sr}(f), \rho, \phi_f, j_f), & \text{if } 0 \leq l(f) < s; \\ k_{d-d(f)}(P^{n-2sr}(f), \rho, \phi_f, j_f), & \text{if } l(f) = s. \end{cases}$$

We have a corresponding first crucial step which is involved in peeling off the factor $P^r_{J(<r)}$ in the left hand side of 25.

Proposition 4.9 *For fixed n and $1 \leq r \leq m$, the alternating sum*

$$\sum_{\substack{f\in F \\ l(f)=s \\ \min\partial C \geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S^z_r(f), \rho, \phi_f, j_f) =$$

$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det^{2s}, j_1) \sum_{f\in F^{n-2sr}(\leq s)} (-1)^{|f|} g_{n,d_2,\rho_2,s,r,j}(f)\right)$$

where $j_1 = j/\gcd(j, q+1, 2s)$, $d_1 + d_2 = d - d(f)$, and $\rho_1\rho_2 = \rho$.

Remark: If $n - 2sr = 0$, then there is a unique $f = (\emptyset, \emptyset, 0) \in F^{n-2sr}(\leq s)$. Moreover, we must set $j_f = 1$ regardless of the value of j in order that our sum may accommodate this degenerate case. Thus

$$\begin{aligned} g_{n,d,\rho,s,r,j}(f) &= k^0_{d-d(f)}(P^{n-2sr}(f), U^{n-2sr}(f), \rho, \phi_f, j_f) \\ &= k^0_d(1, 1, \rho, \det, 1) \\ &= 1 \text{ if and only if } \rho = 1 \text{ and } d = 0. \end{aligned}$$

Proof: The proof is entirely analogous the proof of Proposition 4.7 and consists of considering the same three cases for $f \in F^n(s,r)$:

1. $\partial C = \{r\}$.
2. $\partial C \supsetneq \{r\}$, and
3. $\min(\partial C) > r$.

We omit the proof, but remark on the following difference. For $\tau \in \mathrm{Irr}(V(f))$, τ is linear and hence extendible to $T_{P(f)}(\tau)$. However this does not hold for $\tau \in \mathrm{Irr}(Z(f))$. Rather non-linear $\phi \in \mathrm{Irr}(U(f))$ is extendible to $T_{P(f)}(\phi)$ as $U(f)/(\ker(\phi))$ is an extra special p-group.

We now state the important corollary to Proposition 4.4 which as mentioned will be needed for the inductive case in the proof of Proposition 4.4.

Corollary 4.10 *Assume that Proposition 4.4 holds. Let $r = 0$ in the definition of $h_{n,d,\rho,s,0,j}$ and $g_{n,d,\rho,s,0,j}$. For each $1 \leq s \leq m$ we have the following:*

$$\sum_{f \in F^n(\leq s)} (-1)^{|f|} h_{n,d,\rho,s,0,j}(f) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu) \leq 2s \\ j \mid \gcd(\lambda(\mu),q+1)}} \beta(\mu, a_\rho) \tag{26}$$

$$\sum_{f \in F^n(\leq s)} (-1)^{|f|} g_{n,d,\rho,s,0,j}(f) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu) \leq 2s-1 \\ j \mid \gcd(\lambda(\mu),q+1)}} \beta(\mu, a_\rho) \tag{27}$$

Proof: The assumption that Proposition 4.4 holds implies a number of results. In order of implication these are Proposition 4.1, 17 (via 21), and thus Theorem 1.1, the main theorem of this paper. We proceed by induction on s.

The small case: Let $s = 1$ so that $f \in F^n(\leq 1)$ and f has length 0 or 1.

If $l(f) = 0$ then $f = (\emptyset, \emptyset, 0)$ so $P(f) = \mathrm{U}_n(q)$, $U(f) = 1$, and $d(f) = 0$. The contribution to the left hand side of 27 is $g_{n,d,\rho,s,0,j}(f) = k_d(\mathrm{U}_n(q), \rho, \det, j)$. If $l(f) = 1$ then $f = (J, \{l\}, l)$ so $P(f) = P_J$, $U(f) = U_l$, and $d(f) = 0$. The contribution is $(-1)^{|f|} g_{n,d,\rho,s,0,j}(f) = (-1)^{|J|} k_d(P_J, \rho, \det, j)$.

Hence the left hand side of 27 is

$$\begin{aligned} k_d(\mathrm{U}_n(q), \rho, \det, j) &+ \sum_{\emptyset \neq J \subset I} (-1)^{|J|} k_d(P_J, \rho, \det, j) \\ &= \begin{cases} \beta((n), a_\rho), & \text{if } d = \binom{n}{2} \text{ and } j = 1; \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{28}$$

by 17.

The only partition μ of n with $l(\mu) \leq 1$ is $\mu = (n)$. Moreover $n'((n)) = \binom{n}{2}$ and the only j dividing $\lambda((n))$ is $j = 1$. Hence the right hand side of 27 is equal to the left. Thus 27 holds for $s = 1$.

As for 26 if $l(f) = 0$ then $f = (\emptyset, \emptyset, 0)$ so $P(f) = \mathrm{U}_n(q)$, $U(f) = 1$, and $d(f) = 0$. The contribution to the left hand side of 26 is $h_{n,d,\rho,s,0,j}(f) = k_d(\mathrm{U}_n(q), \rho, \det, j)$. If $l(f) = 1$ then $f = (J, \{l\}, l)$ so $P(f) = P_J$, $U(f) = U_l$, and $d(f) = 0$. The contribution is

$$(-1)^{|J|} k_d^0(P_J, Z_l, \rho, \det, j) = (-1)^{|J|}\big(k_d(P_J, \rho, \det, j) - k_d^1(P_J, Z_l, \rho, \det, j)\big).$$

Hence the left hand side of 26 is given by

$$\begin{aligned} k_d(\mathrm{U}_n(q), \rho, \det, j) + \sum_{\emptyset \neq J \subset I} (-1)^{|J|} k_d(P_J, \rho, \det, j) - \sum_{l(f)=1} (-1)^{|f|} k_d^1(P(f), Z(f), \rho, \phi_f, j_f) \\ = \sum_{f \in F^n(\leq 1)} (-1)^{|f|} g_{n,d,\rho,s,0,j}(f) + \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)=2 \\ j \mid \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho) \\ = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)\leq 2 \\ j \mid \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho) \end{aligned} \tag{29}$$

by 27 and Proposition 4.4. Hence 27 holds for $s = 1$ and the small case is proved.

The inductive case: Let us assume that Corollary 4.10 holds for all $s' < s$. Our first observation is that

$$g_{n,d,\rho,s,r,j}(f) = g_{n-2sr,d,\rho,s,0,j}(f) \text{ and } h_{n,d,\rho,s,r,j}(f) = h_{n-(2s+1)r,d,\rho,s,0,j}(f).$$

Now fixing n and letting $r = 0$ we further observe that for $f \in F^n$ with $l(f) \leq s - 2$

$$g_{n,d,\rho,s,0,j}(f) = h_{n,d,\rho,s-1,0,j}(f) = k_d^0(P^n(f), U^n(f), \rho, \phi_f, j_f).$$

For the rest of the proof we will drop the superscript n and let $P(f)$ denote $P^n(f)$, $U(f)$ denote $U^n(f)$, $Z(f)$ denote $Z^n(f)$, and $V(f)$ denote $V^n(f)$.

Writing $P(f) = Q(f) \times H(f)$ where $Q(f) = \prod_{i=1}^{s-1} P_{J_i(f)}^{+n_i(f)}$ and $H(f) = P_{J_s(f)}^{n_s(f)}$. Let $L(f)$ be a complement to $U(f)$ in $H(f)$. The quotient $P(f)/Z(f) \cong Q(f) \times (L(f) \ltimes V(f))$ and hence

$$k_{d-d(f)}^0(P(f), Z(f), \rho, \phi_f, j_f) = k_{d-d(f)}^0(P(f), U(f), \rho, \phi_f, j_f) + k_{d-d(f)}^1(P(f), V(f), \rho, \phi_f, j_f).$$

For brevity write $g_{d,\rho,s} = g_{n,d,\rho,s,0,j}$ and $h_{d,\rho,s-1} = h_{n,d,\rho,s-1,0,j}$. Then

$$
\begin{aligned}
&\sum_{f\in F^n(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f)\\
&\quad = \sum_{\substack{f\in F^n\\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f),\rho,\phi_f,j_f) + \sum_{\substack{f\in F^n\\ l(f)=s-1}} (-1)^{|f|} k^0_{d-d(f)}(P(f),U(f),\rho,\phi_f,j_f)\\
&\qquad\quad + \sum_{f\in F^n(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f) - \sum_{\substack{f\in F^n\\ l(f)=s-1}} (-1)^{|f|} k^0_{d-d(f)}(P(f),Z(f),\rho,\phi_f,j_f)\\
&\quad = \sum_{\substack{f\in F^n\\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f),\rho,\phi_f,j_f) - \sum_{\substack{f\in F^n\\ l(f)=s-1}} (-1)^{|f|} k^1_{d-d(f)}(P(f),V(f),\rho,\phi_f,j_f)\\
&\qquad\quad + \sum_{f\in F^n(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f)\\
&\quad = - \sum_{\substack{f\in F^n\\ l(f)=s-1}} (-1)^{|f|} k_{d-d(f)}(P(f),S^{nu}(f),\rho,\phi_f,j_f) + \sum_{f\in F^n(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f)\\
&\quad = + \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)=2s-1\\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho) + \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)\leq 2(s-1)=2s-2\\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho)\\
&\quad = \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)\leq 2s-1\\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho)
\end{aligned}
$$

Hence 27 holds at s.

As to 26 notice for $f\in F^n$ $h_{d,\rho,s}(f)=g_{d,\rho,s}(f)$ if $l(f)<s$. Thus

$$
\begin{aligned}
\sum_{f\in F^n(\leq s)} (-1)^{|f|} h_{d,\rho,s}(f) &= \sum_{f\in F^n(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f) - \sum_{\substack{f\in F^n\\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f),\rho,\phi_f,j_f) \\
&\quad + \sum_{\substack{f\in F^n\\ l(f)=s}} (-1)^{|f|} k^0_{d-d(f)}(P(f),Z(f),\rho,\phi_f,j_f) \\
&= \sum_{f\in F^n(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f) - \sum_{\substack{f\in F^n\\ l(f)=s}} (-1)^{|f|} k^1_{d-d(f)}(P(f),Z(f),\rho,\phi_f,j_f) \\
&= \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)\leq 2s-1\\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho) + \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)=2s\\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho) \\
&= \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)\leq 2s\\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho).
\end{aligned}
$$

Hence 26 holds at s and Corollary 4.10 is proved.

Proof of Proposition 4.4: We proceed by induction on m where $n = 2m$ or $n = 2m+1$.

The small cases: Let $m = 0$ so that $n = 1$. Then both sums are empty and hence Proposition 4.4 is vacuously true.

Now let $m = 1$ so that $n = 2$ or $n = 3$. In either case we have $s = r = 1$. For f with $l(f) = 1$ we have $f = (J, \{l\}, l)$ where $\max J = l$. Thus $f = (\{1\}, \{1\}, 1)$ is the only $f \in F^n$ of nonzero length and $P(f) = P_{\{1\}}$ the Borel subgroup of $\mathrm{U}_n(q)$ with $U(f)$ the unipotent radical of $P_{\{1\}}$. Let $L(f)$ be a complement in $P(f)$ to $U(f)$. Notice that $d(f) = 0$ by definition.

Assume $n = 2$ so that we are working in $\mathrm{U}_2(q)$. Then $P(f) \cong \mathrm{GL}_1(q^2) \ltimes F_q$. Under this isomorphism the map ϕ_f is defined $\phi_f(A) = (\det A)^{1-q}$ where $A \in \mathrm{GL}_1(q^2)$. The group $L(f) \cong \mathrm{GL}_1(q^2)$ and $U(f) = Z(f)$ is elementary abelian of order q. We have $V(f) = U(f)/Z(f) = 1$ so that $S^{nu}(f)$ is empty. Thus the left hand side of 24 is

$$-k_{d-d(f)}(P(f), S_1^{nu}(f), \rho, \phi_f, j_f) = 0.$$

This is equal to the right hand side of 24 since there are no partitions $\mu \vdash 2$ of length 3.

As for 25, note the set $S_1^z(f)$ is the collection of nontrivial irreducible characters of F_q on which $\mathrm{GL}_1(q^2) \ltimes F_q$ acts transitively. Indeed let us recall $g \in L(f)$ acts on $\tau \in S_1^z(f)$ via

$$\begin{pmatrix} a & 0 \\ 0 & a^{-q} \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a^q \end{pmatrix} = \begin{pmatrix} 1 & aca^q \\ 0 & 1 \end{pmatrix}.$$

Take any nontrivial $\tau \in \mathrm{Irr}(F_q)$ then $T_{GL_1(q^2)}(\tau) = \mathbb{C}_{q+1}$. Moreover the map ϕ_f restricted to $T_{\mathrm{GL}_1(q^2)}(\tau)$ is defined $\phi_f(A) = (\det A)^2$ for $A \in \mathbb{C}_{q+1}$. We have

$$k_d(\mathrm{GL}_1(q^2) \ltimes F_q, \tau, \rho, \phi_f, j) = k_d(T_{\mathrm{GL}_1(q^2)}(\tau), \rho, \det^2, j')$$

where j' is the least positive integer such that j divides $j' \cdot \gcd(2, q+1)$. Hence assuming that j is a divisor of 2

$$j' = j/\gcd(j, q+1, 2) = \begin{cases} 1, & \text{if } j = 2; \\ 1, & \text{if } j = 1. \end{cases}$$

Then for $j = 1$ or $j = 2$ the left hand side of 25 is given by

$$\begin{aligned} -k_d(P(f), S_1^z(f), \rho, \phi_f, j) &= -k_d(\mathrm{GL}_1(q^2) \ltimes F_q, \tau, \rho, \phi_f, j) \\ &= -k_d(\mathbb{C}_{q+1}, \rho, \det^2, 1) \\ &= \begin{cases} -1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This is certainly equal to the right hand side of 25 since the only partition $\mu \vdash 2$ with $l(\mu) = 2$ is $\mu = (1^2)$ with minimal element 1, $n'((1^2)) = 0$, and $\lambda((1^2)) = 2$ which is divisible by $j = 1$ or $j = 2$.

Assume now $n = 3$ so that we are working in $U_3(q)$. The group

$$P(f) = P_{\{1\}} \cong \big(\mathrm{GL}_1(q^2) \times \mathrm{U}_1(q)\big) \ltimes U_{\{1\}}.$$

Under this isomorphism the map ϕ_f is defined $\phi_f(A, B) = (\det A)^{1-q} \det B$ where $A \in \mathrm{GL}_1(q^2)$ and $B \in \mathrm{U}_1(q)$. In this case $U(f) = U_{\{1\}}$ is no longer equal to its center. Let $L(f)$ be a complement to $U(f)$ in $P(f)$.

The quotient $V(f) \cong M_{1,1}(F_{q^2})$ and $S_1^{nu}(f)$ has one orbit under $P_{\{1\}}$. Take $\tau \in S_1^{nu}(f)$ then $T_{L(f)}(\tau) = \mathrm{U}_1(q) \cong \mathbb{C}_{q+1}$. Moreover the map ϕ_f restricted to $T_{L(f)}(\tau)$ is defined $\phi_f(B) = (\det B)^3$ for $B \in \mathrm{U}_1(q)$

Then for $j = 1$ or $j = 3$ the left hand side of 24 is given by

$$\begin{aligned} -k_d(P(f), S_1^{nu}(f), \rho, \phi_f, j) &= -k_d(P(f), \tau, \rho, \phi_f, j) \\ &= -k_d(T_{L(f)}(\tau), \rho, \phi_f|, j/\gcd(j, 3, q+1)) \\ &= -k_d(\mathbb{C}_{q+1}, \rho, \det^3, 1) \\ &= \begin{cases} -1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This is certainly equal to the right hand side of 24 since the only partition $\mu \vdash 3$ with $l(\mu) = 2(1) + 1 = 3$ is $\mu = (1^3)$ with minimal element 1, $n'((1^3)) = 0$, and $\lambda((1^3)) = 3$ which is divisible by $j = 1$ or $j = 3$.

As for 25, $S_1^z(f)$ is the set of non-trivial characters of $Z(f)$ on which $P(f)$ acts transitively. Take $\tau \in S_1^z(f)$. Then there exists a unique $\psi \in \mathrm{Irr}(U(f))$ lying over τ with $\psi(1) = q$. We have

$$T_P(\tau) = T_P(\psi) = \big(\mathrm{U}_1(q) \times \mathrm{U}_1(q)\big) \ltimes U_{\{1\}}$$

and q does not divide $|P|/|T_P(\psi)|$. The map ϕ_f restricted to $T_P(\psi)$ is defined $\phi_f(A,B) = (\det A)^2(\det B)$ for A in the first factor $\mathrm{U}_1(q)$ and B in the second factor $\mathrm{U}_1(q)$. Also notice that

$$\left|P(f)/T_P(\psi)\ker(\phi_f)\right| = 1.$$

Hence the left hand side of 24 is given by

$$\begin{aligned}
-k_d(P(f), S_1^z(f), \rho, \phi_f, j) &= -k_d(P(f), \tau, \rho, \phi_f, j)\\
&= -k_d(P(f), \psi, \rho, \phi_f, j)\\
&= -k_{d-1}(T_{P(f)}(\psi)/U_{\{1\}}, \rho, \phi_f|, j)\\
&= -k_{d-1}(\mathbb{C}_{q+1}\times\mathbb{C}_{q+1}, \rho, \det{}^2\cdot\det, j)\\
&= -\sum_{\rho_1\rho_2=\rho}\sum_{d_1+d_2=d-1} k_{d_1}(\mathbb{C}_{q+1}, \rho_1, \det{}^2, j/\gcd(j,2,q+1))k_{d_2}(\mathbb{C}_{q+1}, \rho_2, \det, j)\\
&= \begin{cases} -(q+1), & \text{if } d=1 \text{ and } j=1;\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

This is certainly equal to the right hand side of 25 since the only partition $\mu \vdash 3$ of length 2 is $\mu = (2,1)$ with $n'((2,1)) = 1$, $\min((2,1)) = 1$, and $\lambda((2,1)) = 1$. Indeed $-\beta((2,1), a_\rho) = -(q+1)$.

The inductive case: We assume Proposition 4.4 holds for all $m' < m$ where $m \geq 2$ and $n = 2m$ or $n = 2m+1$. Recall that this assumption implies that for all such m', not only does Corollary 4.10 hold at m', but also Proposition 4.1, 17, and hence Theorem 1.1, the main theorem of this paper, hold at m'.

Recall our observation in the proof of 4.10 that

$$h_{n,d,\rho,s,r,j}(f) = h_{n-(2s+1)r,d,\rho,s,0,j}(f).$$

Moreover since $1 \leq r < n$ by induction assumption we have

$$\sum_{J\subseteq[r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det{}^{2s+1}, j_1) = \begin{cases} \beta((r), a_\rho), & \text{if } d_1 = \binom{r}{2} \text{ and } j_1 = 1;\\ 0, & \text{otherwise.}\end{cases}$$

Notice that if $j_1 = 1$ we have $j = \gcd(j, 2s+1, q+1)$ in the statement of Proposition

4.7. Then

$$\sum_{\substack{f\in F\\ l(f)=s\\ \min\partial C\geq r}} (-1)^{|f|}k_{d-d(f)}(P(f),S_r^{nu}(f),\rho,\phi_f,j_f) =$$

$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]}(-1)^{|J|}k_{d_1}(P_J,\rho_1,\det^{2s+1},j_1)\sum_{f\in F^{n-(2s+1)r}(\leq s)}(-1)^{|f|}h_{n,d_2,\rho_2,s,r,j}(f)\right)$$

$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]}(-1)^{|J|}k_{d_1}(P_J,\rho_1,\det^{2s+1},j_1)\sum_{f\in F^{n-(2s+1)r}(\leq s)}(-1)^{|f|}h_{n-(2s+1)r,d_2,\rho_2,s,0,j}(f)\right)$$

$$= -\sum_{\rho_1\rho_2=\rho}\left(\beta((r),a_{\rho_1})\sum_{\substack{\mu\vdash(n-(2s+1)r)\\ n'(\mu)=d_2\\ l(\mu)\leq 2s\\ j|\gcd(2s+1,q+1,\lambda(\mu))}}\beta(\mu,a_{\rho_2})\right)$$

$$= -\sum_{\substack{\tilde\mu\vdash(n)\\ n'(\tilde\mu)=d_2\\ \min(\tilde\mu)=r\\ l(\tilde\mu)\leq 2s+1\\ j|\gcd(q+1,\lambda(\tilde\mu))}}\beta(\tilde\mu,a_{\rho_2})$$

where for fixed $\mu=(a_1^{l_1},a_2^{l_2},\ldots,a_{\delta(\mu)}^{l_{\delta(\mu)}})\vdash(n-(2s+1)r)$ the corresponding partition of n is given by

$$\tilde\mu=(r^{2s+1})+\mu=((a_1+r)^{l_1},(a_2+r)^{l_2},\ldots,(a_{\delta(\mu)}+r)^{l_{\delta(\mu)}},r^e)\vdash n.$$

and $\sum_{i=1}^{\delta(\mu)}l_i+e=2s+1$ so j divides e i.e. j divides $\lambda(\tilde\mu)$.

Recall our observation in the proof of 4.10 that

$$g_{n,d,\rho,s,r,j}(f)=g_{n-2sr,d,\rho,s,0,j}(f).$$

Moreover since $1\leq r<n$ by induction assumption we have

$$\sum_{J\subseteq[r/2]}(-1)^{|J|}k_{d_1}(P_J,\rho_1,\det^{2s},j_1)=\begin{cases}\beta((r),a_\rho), & \text{if } d_1=\binom{r}{2} \text{ and } j_1=1;\\ 0, & \text{otherwise.}\end{cases}$$

Notice that if $j_1=1$ we have $j=\gcd(j,2s,q+1)$ in the statement of Proposition 4.9.

Then

$$\sum_{\substack{f\in F\\ l(f)=s\\ \min\partial C\geq r}} (-1)^{|f|}k_{d-d(f)}(P(f),S_r^z(f),\rho,\phi_f,j_f) =$$

$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]}(-1)^{|J|}k_{d_1}(P_J,\rho_1,\det^{2s},j_1)\sum_{f\in F^{n-2sr}(\leq s)}(-1)^{|f|}g_{n,d_2,\rho_2,s,r,j}(f)\right)$$
$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]}(-1)^{|J|}k_{d_1}(P_J,\rho_1,\det^{2s},j_1)\sum_{f\in F^{n-2sr}(\leq s)}(-1)^{|f|}g_{n-2sr,d_2,\rho_2,s,0,j}(f)\right)$$
$$=-\sum_{\rho_1\rho_2=\rho}\left(\beta((r),a_{\rho_1})\sum_{\substack{\mu\vdash(n-2sr)\\ n'(\mu)=d_2\\ l(\mu)\leq 2s-1\\ j|\gcd(2s,q+1,\lambda(\mu))}}\beta(\mu,a_{\rho_2})\right)$$
$$=-\sum_{\substack{\tilde{\mu}\vdash(n)\\ n'(\tilde{\mu})=d_2\\ \min(\tilde{\mu})=r\\ l(\tilde{\mu})\leq 2s\\ j|\gcd(q+1,\lambda(\tilde{\mu}))}}\beta(\tilde{\mu},a_{\rho_2})$$

where for fixed $\mu=(a_1^{l_1},a_2^{l_2},\dots,a_{\delta(\mu)}^{l_{\delta(\mu)}})\vdash(n-2sr)$ the corresponding partition of n is given by

$$\tilde{\mu}=(r^{2s})+\mu=((a_1+r)^{l_1},(a_2+r)^{l_2},\dots,(a_{\delta(\mu)}+r)^{l_{\delta(\mu)}},r^e)\vdash n.$$

and $\sum_{i=1}^{\delta(\mu)}l_i+e=2s$ so j divides e i.e. j divides $\lambda(\tilde{\mu})$.

And we are done, Proposition 4.4 is proved. Thus Proposition 4.1 also holds, so too does 17 (via 21), and thus the main result of this paper Theorem 1.1 is proved.

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