A Note on a Special Kind of Singular Points of An Algebraic Variety

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Let $F(x) = 0$ be a polynomial system with n equations in n unknowns, where $x \in C^n$. Homotopy continuation method can be used to find all isolated solutions of $F(x) = 0$ and to detect connected components as well. Let $H(x,t)$ be a polynomial system, where $t \in [0,1]$ is the parameter, such that $H(x,0) = F(x)$ and the coefficients of $H(x,t)$ are in general position for $t \neq 0$. It is well known that if V is a connected component of solution set of $F(x) = 0$, then there is at least one solution curve of $H(x,t) = 0$ converging to a point $p \in V$. In general when a different homotopy is applied, the point p will be different. However, in some special cases, some points p are invariant with respect to different homotopies. So they can be mistaken as isolated points of $V(F)$.

In this paper, we prove the following

Theorem 1 *Let $F(x) = 0$ be a polynomial system with n equations in n unknowns. Suppose that V is a k-dimensional connected component of $V(F)$ and $p \in V$, where $k > 1$. In addition, V has no "bifurcation" at p, $rank(D_x F(p)) \leq n-k-1$, $rank(D_x F(x)) = n-k$ for $x \in V$ and "near" p. Let $H(x,t)$ be a polynomial system such that $H(x,0) = F(x)$ and the coefficients of $H(x,t)$ are in general position for $t \neq 0$. Then there is at least one solution curve of $H(x,t) = 0$, $(x,t) = (x(t),t)$ with $(x_1,...,x_n)$ not contained in V so that $(x,t) \to (p,0)$ as $t \to 0$.*

Example 1 *Let*

$$F(x,y) := \begin{cases} xy &= 0, \\ y^2 &= 0. \end{cases}$$

Then $V = \{y = 0\}$ and $p = (0,0)$. Let

$$H(x,y,t) := \begin{cases} xy &- c_1 t = 0, \\ y^2 &- c_2 t = 0. \end{cases}$$

Then the curves $y = \pm\sqrt{c_2 t}, x = \pm c_1\sqrt{t/c_2}$ converge to p as $t \to 0$.
