The Challenges of Numerical Polynomial Algebra

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Numerical Polynomial Algebra (NPA) involves the consideration of algebraic problems with polynomial data of limited accuracy and their computational treatment in floating-point arithmetic. It is thus analogous to Numerical Linear Algebra (NLA) which deals with linear functions under the same assumptions. NLA has been developed over the past 40 years; it has become the computational basis of almost all of scientific computing. NPA, on the other hand, has remained at a low level of development so that various real-world tasks modelled with polynomials cannot be solved adequately.

For the use with inexact data, we must modify many algebraic concepts because they depend discontinuously on their data (e.g. the multiplicity of a polynomial zero); here, continuity refers to the natural topology of tuples of real or complex numbers. Moreover, algebraic problems are often overdetermined but consistent; they may have no solution after an arbitrarily small data change. In such cases, we must find the closest data which permit a solution and decide if these data fall within the indeterminacy of our specified problem. Thus, the concept of an exact solution turns into that of a *valid* solution. How can such problems be reasonably treated in an floating-point environment? After a general exposition of these difficulties, we review some important critical problem categories and present numerical examples.

Multiple zeros and zero clusters: With data of limited accuracy, zero clusters and multiple zeros are interrelated concepts. In the multidimensional situation, the differential structure of the underlying multiple zero determines the statistical distribution of the zeros in an associated cluster.

“Approximate” Groebner bases: Groebner bases in n-space usually consist of more than n polynomials which are not independent but satisfy syzygy relations. When computed numerically, such Groebner bases do not satisfy the syzygies exactly; therefore they are slightly inconsistent. We will show why this deficiency is acceptable in many applications and how it can be removed if necessary.

Singular multivariate polynomial systems: The existence of a positive-dimensional zero manifold may be a singular situation, with the manifold disappearing at the tiniest change of the data. We will see how this discontinuous phenomenon may be analytically understood as a smooth transition and discuss the computational consequences.

Factorization of a multivariate polynomial: Factorizability implies that the manifold defined by the polynomial consists of separate components. An empirical polynomial will "never" factor, but there may be one with nearly the same data which does. Then the partitioned manifold will be the more natural representation of the underlying model.

A short review of the state-of-the-art will conclude the talk.