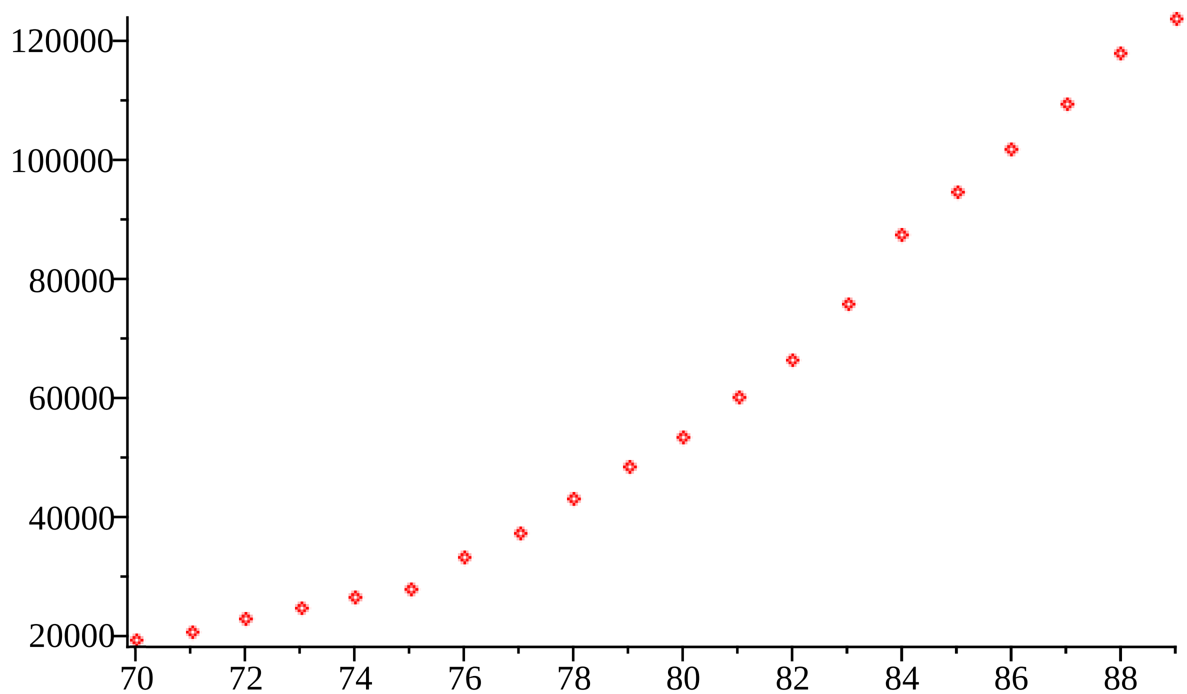
```
> restart:
  x := Vector([seq(k, k=70..89)]):
> y := Vector([19550,20700,23210,24980,26620,27900,33300,37440,
  43330,48780,53550,60430,66580,75850,87820,94750,102140,109650,
  118050,123930]):
> x[1..10],y[1..10],x[11..20],y[11..20];
```

$$\begin{bmatrix}70\\71\\72\\73\\74\\75\\76\\77\\78\\79\end{bmatrix}, \begin{bmatrix}19550\\20700\\23210\\24980\\26620\\27900\\33300\\37440\\43330\\48780\end{bmatrix}, \begin{bmatrix}80\\81\\82\\83\\84\\85\\86\\87\\88\\89\end{bmatrix}, \begin{bmatrix}53550\\60430\\66580\\75850\\87820\\94750\\102140\\109650\\118050\\123930\end{bmatrix} \quad \textbf{(1)}$$

```
> plot([seq([x[k],y[k]],k=1..20)],style=point);
```



Assume linear model

$$y = a + bx$$

Q: $a = ?$ $b = ?$

$$a + bx_1 = y_1$$
$$a + bx_2 = y_2$$
$$a + bx_3 = y_3$$
$$\dots$$

In matrix form

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_{20} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{20} \end{bmatrix}$$

$\rightarrow$ Least square solution

```
> n := 20:
  A := Matrix(n,2):
```

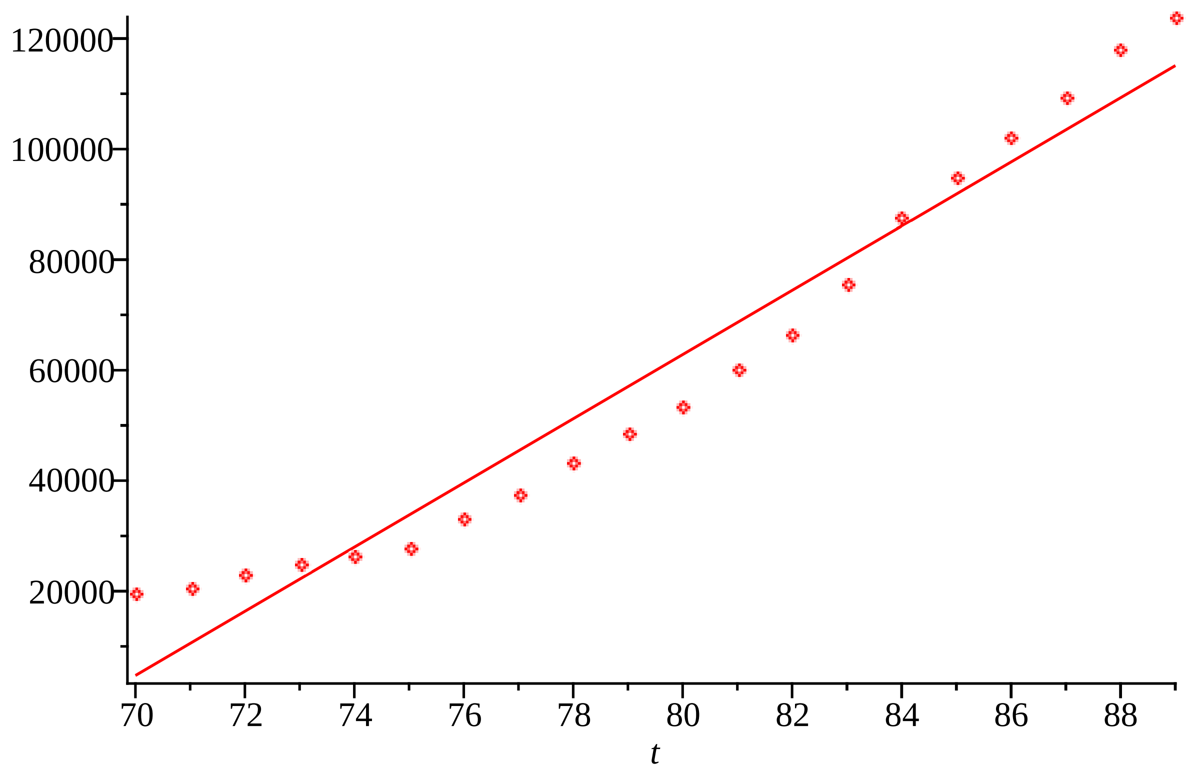
```
A[1..-1,1] := Vector([seq(1,k=1..n)]):
A[1..-1,2] := x:
A[1..10,1..2],A[11..n,1..2];
```

$$\begin{bmatrix} 1 & 70 \\ 1 & 71 \\ 1 & 72 \\ 1 & 73 \\ 1 & 74 \\ 1 & 75 \\ 1 & 76 \\ 1 & 77 \\ 1 & 78 \\ 1 & 79 \end{bmatrix}, \begin{bmatrix} 1 & 80 \\ 1 & 81 \\ 1 & 82 \\ 1 & 83 \\ 1 & 84 \\ 1 & 85 \\ 1 & 86 \\ 1 & 87 \\ 1 & 88 \\ 1 & 89 \end{bmatrix} \tag{2}$$

```
> u := LinearAlgebra:-LeastSquares(A,1.0*y);
```

$$u := \begin{bmatrix} -4.01900052631585277\ 10^{5} \\ 5809.15789473692166 \end{bmatrix} \tag{3}$$

```
> lplot := plot(u[1]+u[2]*t,t=70..89):
  pplot := plot([seq([x[k],y[k]],k=1..20)],style=point):
> plots[display]({lplot,pplot});
```



```
> SSres := LinearAlgebra:-Norm(A.u-y,2)^2;
```

$$SSres := 1.105019548\ 10^{9} \tag{4}$$

```
> ybar := add(y[k],k=1..20)/20;
```

$$ybar := 59928 \tag{5}$$

```
> SStol := add((y[k]-ybar)^2,k=1..20);
```

$$SStol := 23546319320 \tag{6}$$

```
> R2 := (SStol-SSres)/SStol;
```

$$R2 := 0.9530703914 \tag{7}$$

```
>
```

Try Quadratic model

$$y = a + bx + cx^2$$

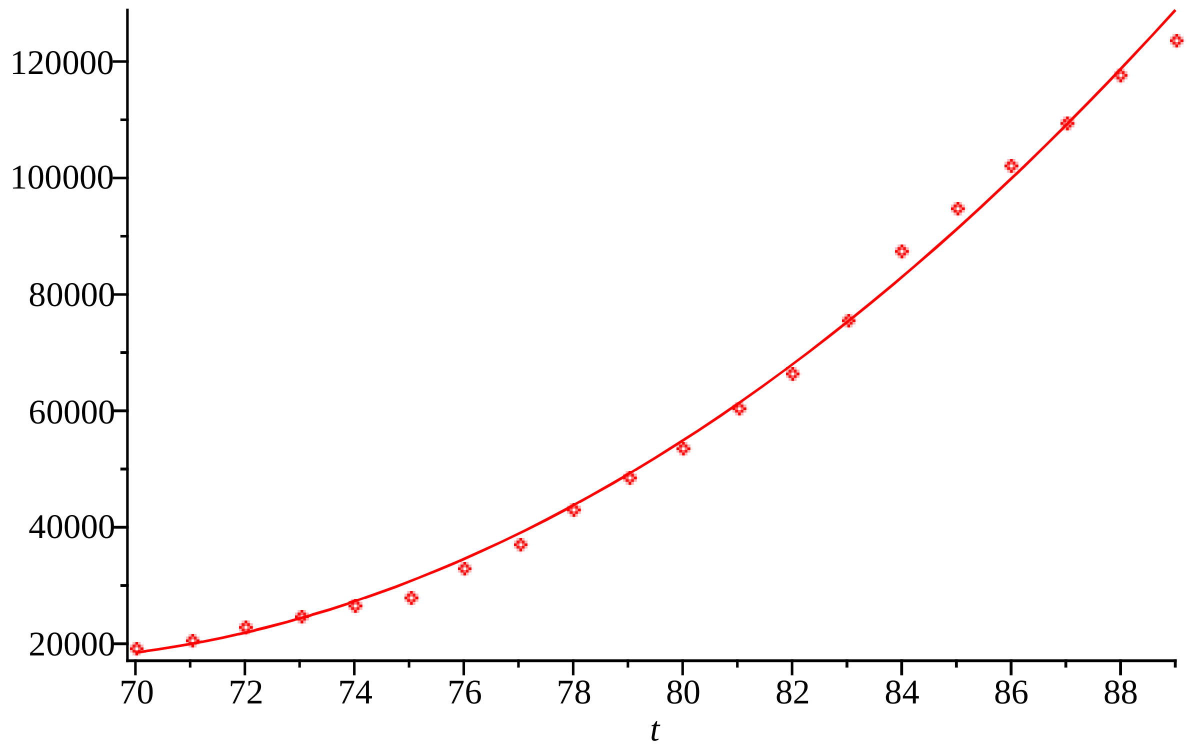
$$\begin{cases} a + bx_1 + cx_1^2 = y_1 \\ \vdots \end{cases}$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_{20} & x_{20}^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{20} \end{bmatrix}$$

```
>
>
> A := Matrix(n,3):
  for k from 1 to n do
     A[k,1] := 1.0:
     A[k,2] := x[k]:
     A[k,3] := x[k]^2:
  end do:
> v := LinearAlgebra:-LeastSquares(A,y);
```

$$v := \begin{bmatrix} 1.11178579949959973\ 10^6 \\ -32472.3834586682278 \\ 240.764411027705080 \end{bmatrix} \tag{8}$$

```
> qplot := plot( v[1]+v[2]*t+v[3]*t^2, t= 70..89 ):
> plots[display]({qplot,pplot});
```




```
> LinearAlgebra:-Norm(A.v-y,2);
```

$$9345.69901211582874 \quad \textbf{(9)}$$

```
>
```

Try exponential model

$$y = a e^{bx}$$

linearize this model

$$\ln y = \ln a \cdot e^{bx} = \ln a + \ln e^{bx}$$

$$= \ln a + bx$$

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_{20} \end{bmatrix} \begin{bmatrix} \ln a \\ b \end{bmatrix} = \begin{bmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_{20} \end{bmatrix}$$

```
> A := Matrix(20,2):
  d := Vector(20):
  for k from 1 to n do
     A[k,1] := 1.0:
     A[k,2] := x[k]:
     d[k] := evalf(ln(y[k])):
  end do:
```

```
> w := LinearAlgebra:-LeastSquares(A,d);
```

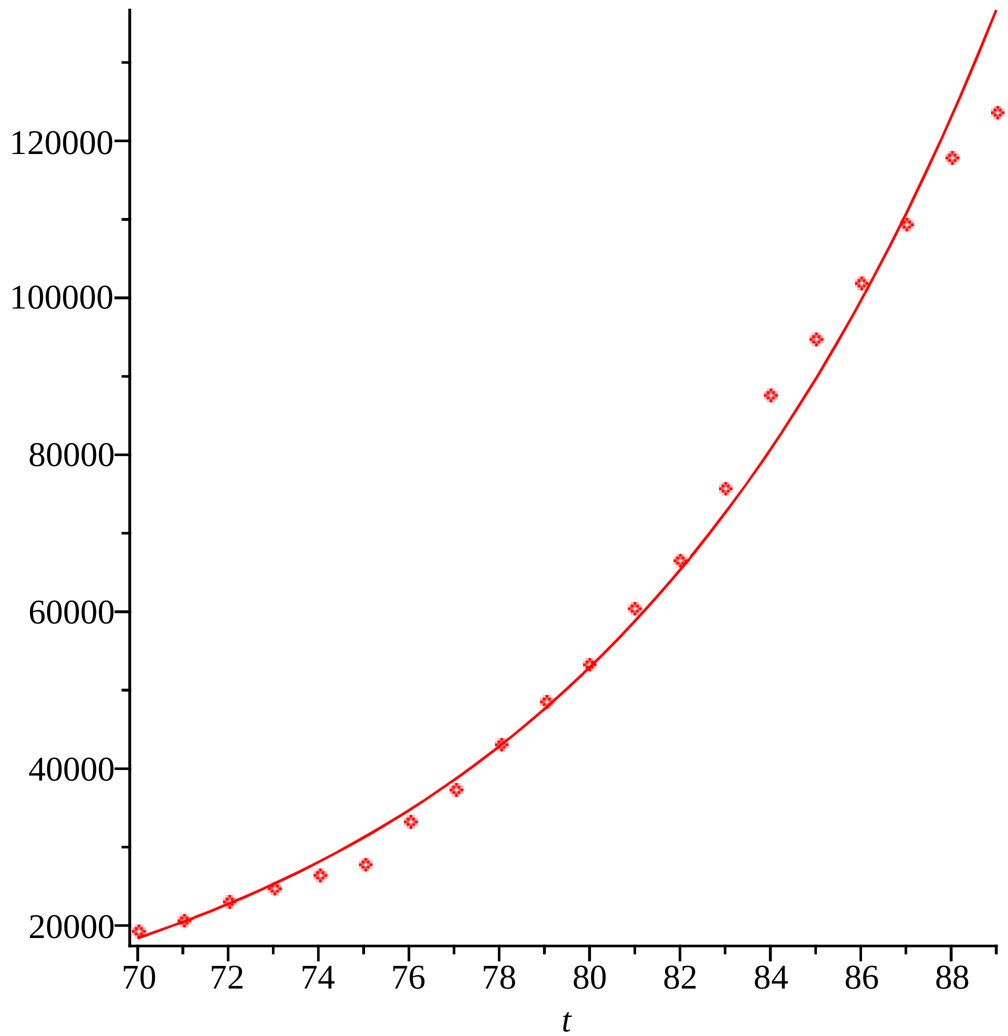
$$w := \begin{bmatrix} 2.43693586980225474 \\ 0.105488783407518812 \end{bmatrix} \tag{10}$$

```
> a := exp(w[1]); b := w[2];
```

$$a := 11.43793966$$
$$b := 0.105488783407518812 \tag{11}$$

```
> eplot := plot( a*exp(b*t), t= 70..89):
```

```
> plots[display]({eplot,pplot});
```




```
> z := Vector([seq( a*exp(b*x[k]), k=1..n )]):
> SSres := 0:
  for k from 1 to 20 do
     SSres := SSres + (z[k]-y[k])^2
  end do:
> SStot := 0:
  for k from 1 to 20 do
     SStot := SStot + (y[k]-ybar)^2
  end do:
> R2 := (SStot-SSres)/SStol;
```

$$R2 := 0.9869561214 \qquad (12)$$

```
>
>
> LinearAlgebra:-Norm(z-y,2);
```

$$17525.2770012585388 \qquad (13)$$

```
>
```

$$y = \frac{ax}{1+bx} \quad \text{not linear}$$

$$(1+bx)\,y = ax$$

$$ax - bxy = y$$

$$ax_1 + bx_1y_1 = y_1$$

$$\vdots$$

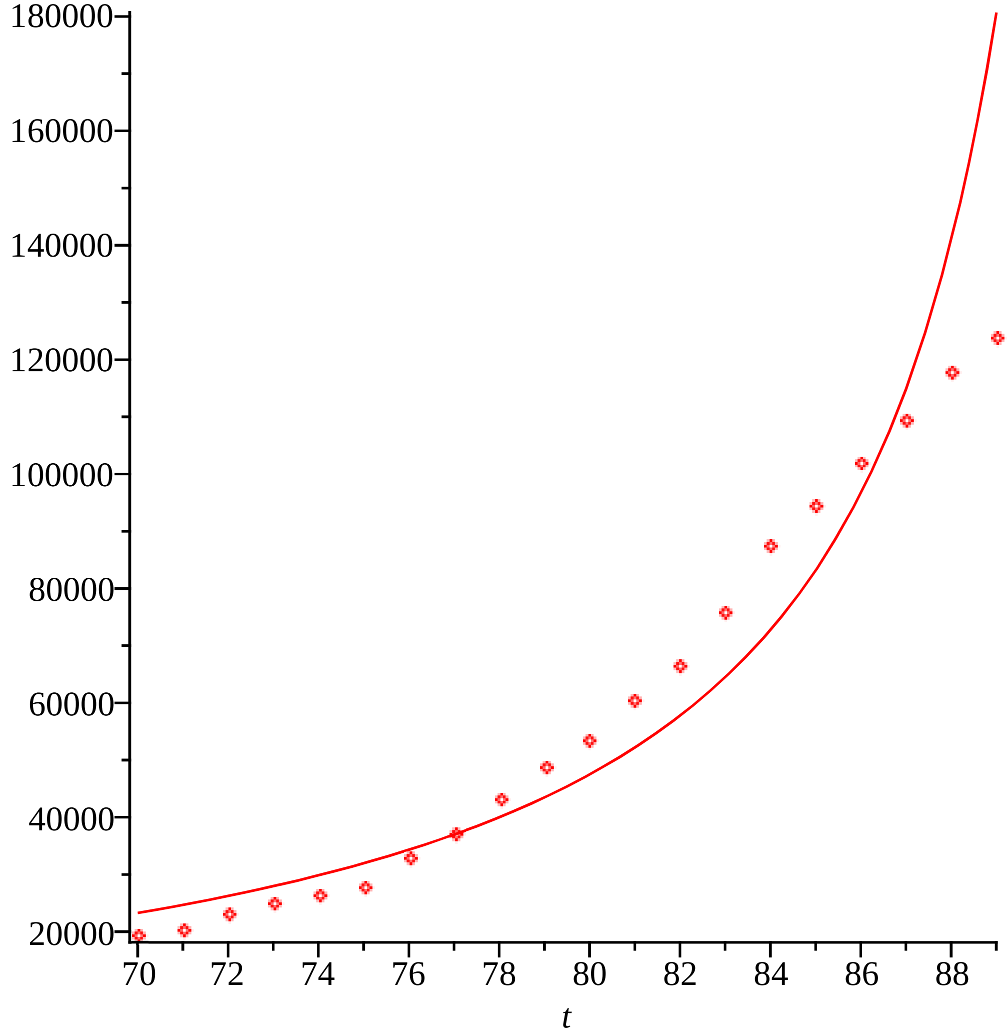
$$\begin{bmatrix} x_1, & -x_1y_1 \\ x_2, & -x_2y_2 \\ \vdots & \vdots \\ x_n, & -x_ny_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

```
> A := Matrix(n,2):
  for k from 1 to n do
     A[k,1] := 1.0*x[k]:
     A[k,2] := -x[k]*y[k]:
  end do:
> p := LinearAlgebra:-LeastSquares(A,y);
```

(14)

$$p := \begin{bmatrix} 81.4332038737884574 \\ -0.0107853032506150510 \end{bmatrix} \qquad (14)$$

```
> rplot := plot( p[1]*t/(1+p[2]*t), t = 70..89):
> plots[display]({rplot,pplot});
```




```
> z := Vector([seq( p[1]*x[k]/(1+p[2]*x[k]),k=1..n)]):
> LinearAlgebra:-Norm(z-y,2);
```

$$67725.3261260830186 \qquad (15)$$

```
>
```

Power model

$$y = a x^b$$