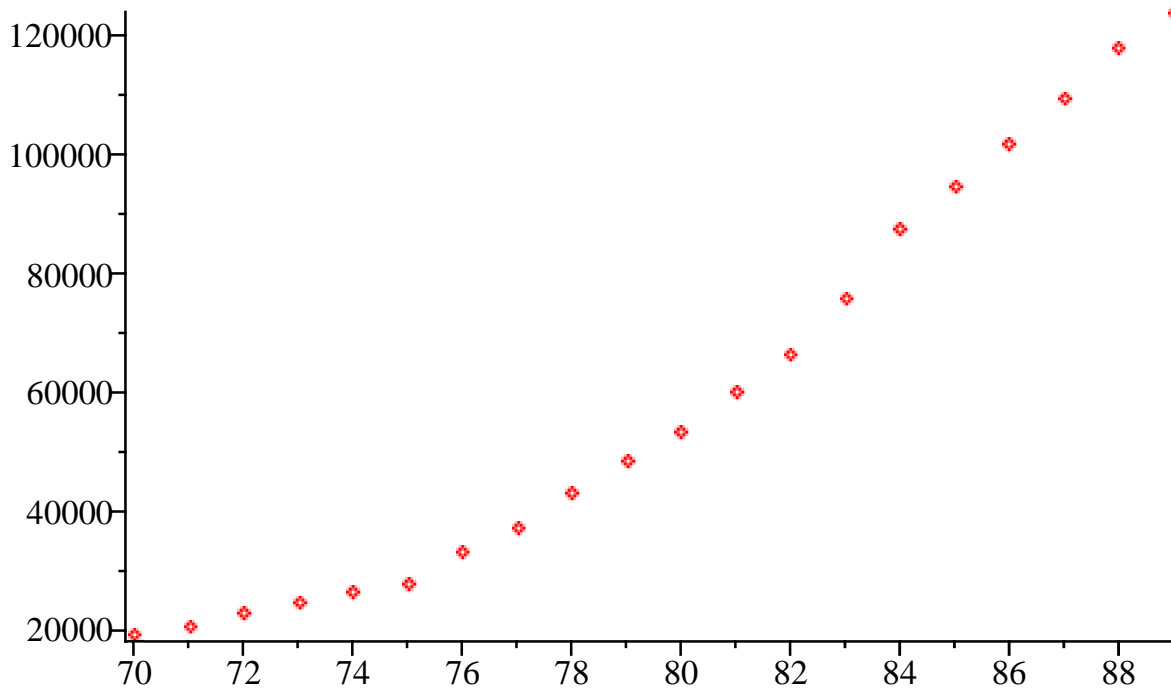


```
> restart:  
x := Vector([seq(k, k=70..89)]):  
> y := Vector([19550,20700,23210,24980,26620,27900,33300,37440,  
43330,48780,53550,60430,66580,75850,87820,94750,102140,109650,  
118050,123930]):  
> x[1..10],y[1..10],x[11..20],y[11..20];
```

70	19550	80	53550
71	20700	81	60430
72	23210	82	66580
73	24980	83	75850
74	26620	84	87820
75	27900	85	94750
76	33300	86	102140
77	37440	87	109650
78	43330	88	118050
79	48780	89	123930

(1)

```
> plot([seq([x[k],y[k]],k=1..20)],style=point);
```



Assume linear model

$$y = a + bx$$

Q: $a = ?$ $b = ?$

$$a + bx_1 = y_1$$

$$a + bx_2 = y_2$$

$$a + bx_3 = y_3$$

...

In matrix form

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_{20} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{20} \end{bmatrix}$$

→ Least square solution

```
> n := 20:  
A := Matrix(n,2):
```

```

A[1..-1,1] := Vector([seq(1,k=1..n)]):
A[1..-1,2] := x:
A[1..10,1..2],A[11..n,1..2];

```

$$\begin{bmatrix} 1 & 70 \\ 1 & 71 \\ 1 & 72 \\ 1 & 73 \\ 1 & 74 \\ 1 & 75 \\ 1 & 76 \\ 1 & 77 \\ 1 & 78 \\ 1 & 79 \end{bmatrix}, \begin{bmatrix} 1 & 80 \\ 1 & 81 \\ 1 & 82 \\ 1 & 83 \\ 1 & 84 \\ 1 & 85 \\ 1 & 86 \\ 1 & 87 \\ 1 & 88 \\ 1 & 89 \end{bmatrix}$$

(2)

```

> u := LinearAlgebra:-LeastSquares(A,1.0*y);

```

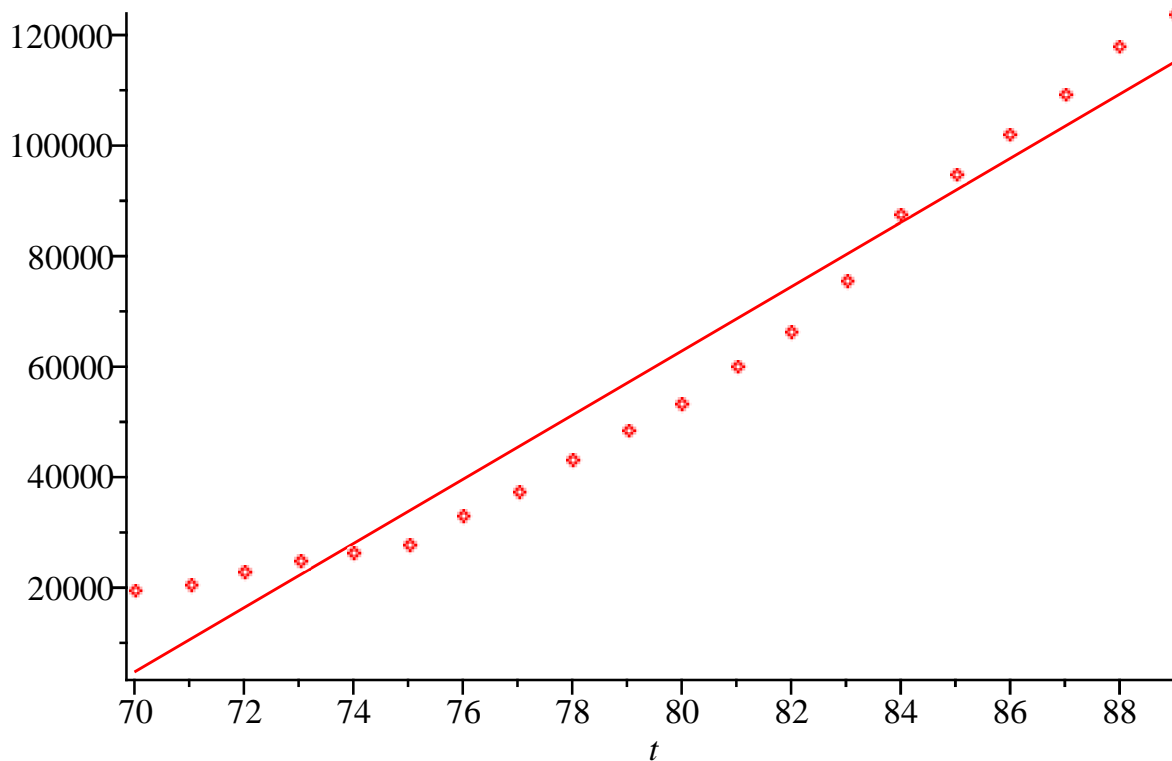
$$u := \begin{bmatrix} -4.01900052631585277 \cdot 10^5 \\ 5809.15789473692166 \end{bmatrix}$$

(3)

```

> lplot := plot(u[1]+u[2]*t,t=70..89):
ppplot := plot([seq([x[k],y[k]],k=1..20)],style=point):
> plots[display]({lplot,ppplot});

```



> LinearAlgebra:-Norm(A.u-y,2);

33241.8343119186611

(4)

Try Quadratic model

$$y = a + bx + cx^2$$

$$\left\{ \begin{array}{l} a + bx_1 + cx_1^2 = y_1 \\ \vdots \\ \vdots \\ \vdots \end{array} \right.$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

>

>

```
> A := Matrix(n,3):  
for k from 1 to n do  
  A[k,1] := 1.0:  
  A[k,2] := x[k]:  
  A[k,3] := x[k]^2:
```

```
end do:
```

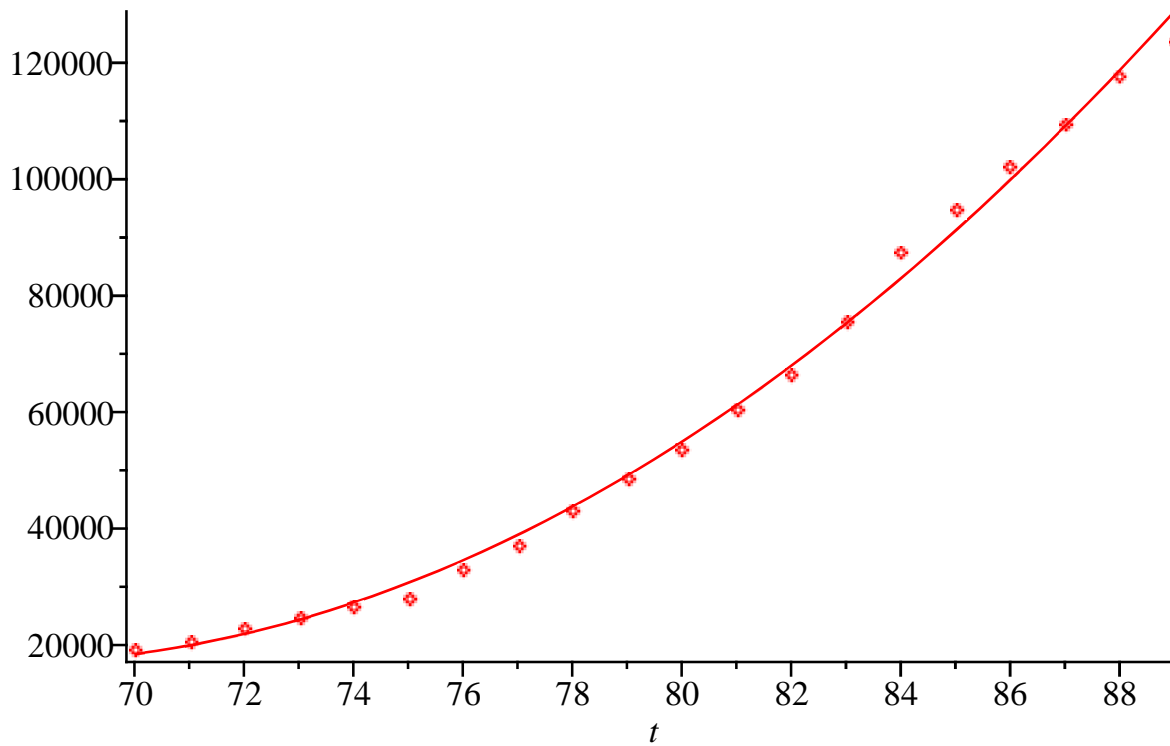
```
> v := LinearAlgebra:-LeastSquares(A,y);
```

$$v := \begin{bmatrix} 1.11178579949959973 \cdot 10^6 \\ -32472.3834586682278 \\ 240.764411027705080 \end{bmatrix}$$

(5)

```
> qplot := plot( v[1]+v[2]*t+v[3]*t^2, t= 70..89 );
```

```
> plots[display]({qplot,pplot});
```



```
> LinearAlgebra:-Norm(A.v-y,2);
```

```
9345.69901211582874
```

(6)

```
>
```

Try exponential model

$$y = a e^{bx}$$

linearize this model

$$\begin{aligned} \ln y &= \ln a \cdot e^{bx} = \ln a + \ln e^{bx} \\ &= \ln a + bx \end{aligned}$$

$$\begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_{20} \end{bmatrix} \begin{bmatrix} \ln a \\ b \end{bmatrix} = \begin{bmatrix} \ln y_1 \\ \vdots \\ \ln y_{20} \end{bmatrix}$$

```
> A := Matrix(20,2):  
d := Vector(20):  
for k from 1 to n do  
  A[k,1] := 1.0:
```

```

A[k,2] := x[k]:
d[k] := evalf(ln(y[k])):
end do:

```

```

> w := LinearAlgebra:-LeastSquares(A,b);

```

$$w := \begin{bmatrix} 2.43693586980225474 \\ 0.105488783407518812 \end{bmatrix}$$

(7)

```

> a := exp(w[1]); b := w[2];

```

$$a := 11.43793966$$

$$b := 0.105488783407518812$$

(8)

```

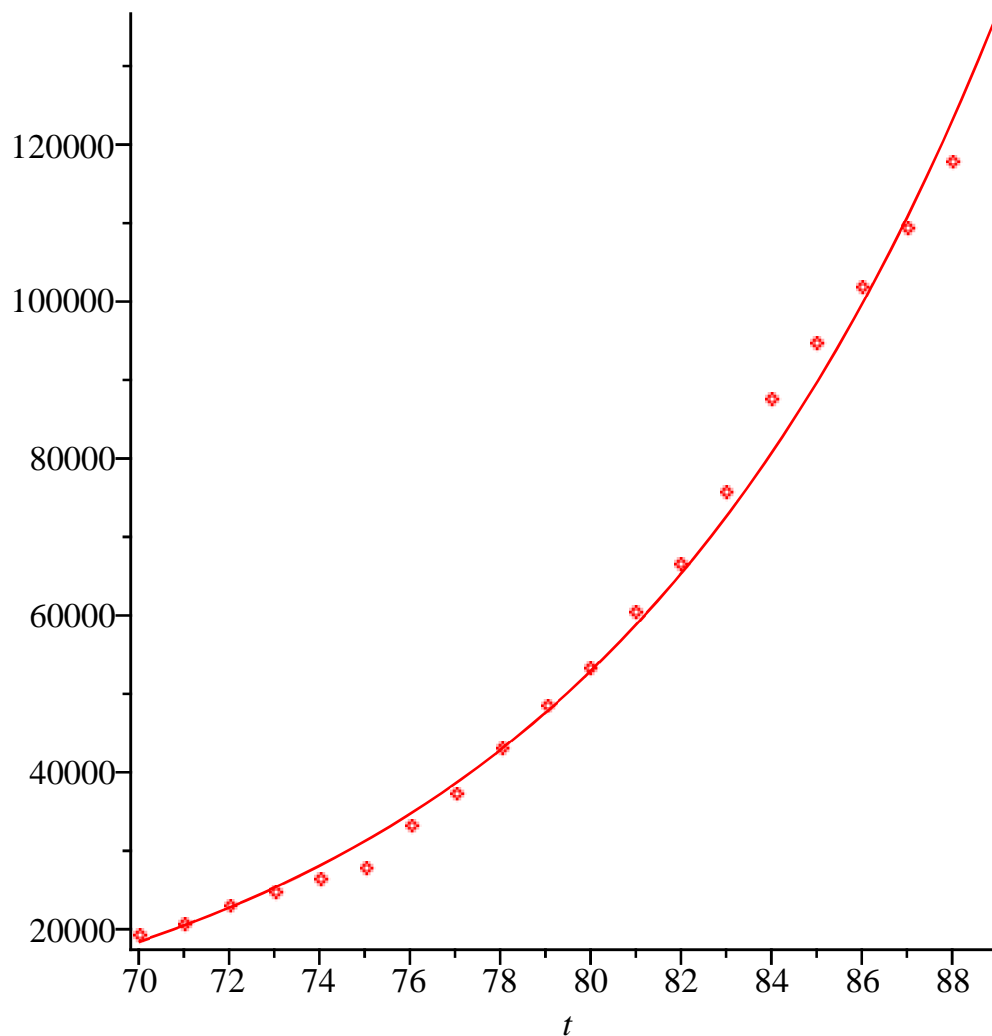
> eplot := plot( a*exp(b*t), t= 70..89):

```

```

> plots[display]({eplot,pplot});

```



```

> z := Vector([seq( a*exp(b*x[k]), k=1..n )]):

```

```

> LinearAlgebra:-Norm(z-y,2);

```

$$17525.2770012585388$$

(9)

```

>

```

$$y = \frac{ax}{1+bx} \quad \text{not linear}$$

$$(1+bx)y = ax$$

$$ax - bxy = y$$

$$ax_1 + b x_1 y_1 = b_1$$

i

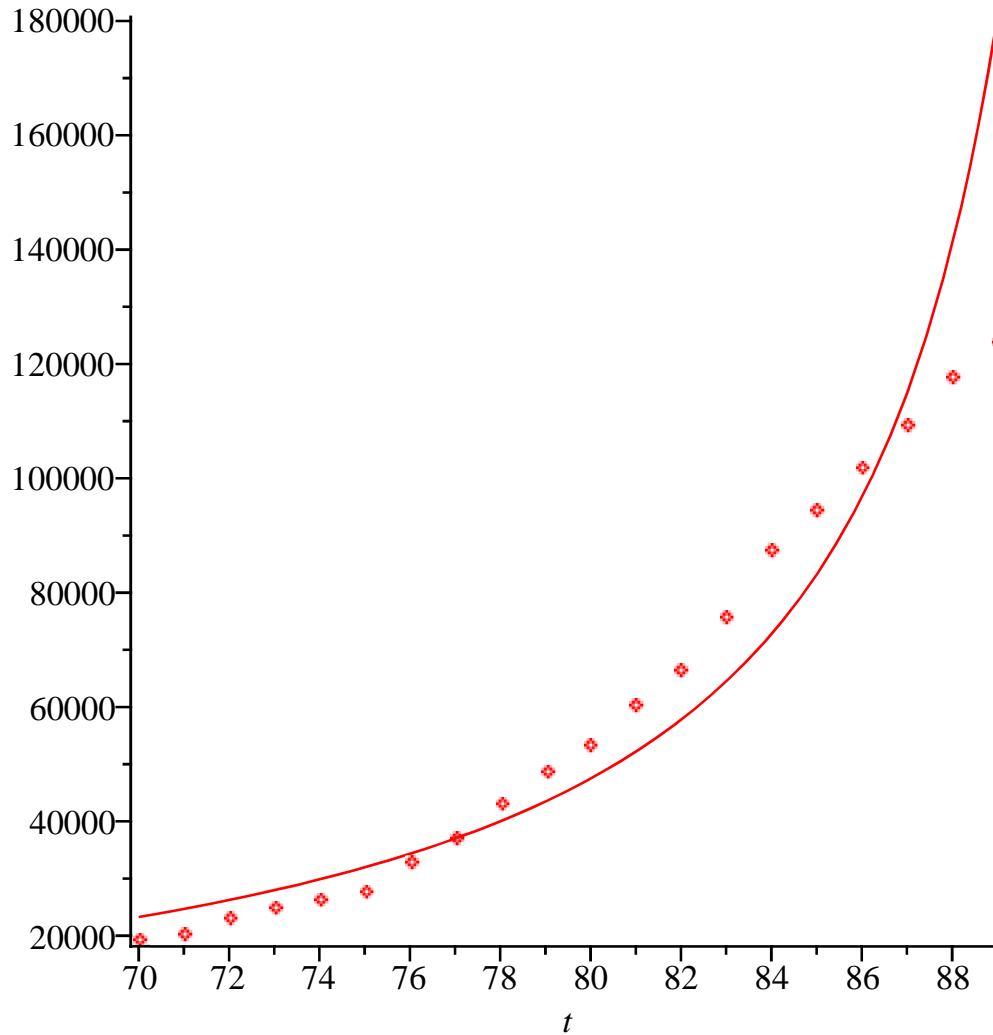
$$\begin{bmatrix} x_1 & -x_1 y_1 \\ x_2 & -x_2 y_2 \\ \vdots & \vdots \\ x_n & -x_n y_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

```
> A := Matrix(n,2):  
  for k from 1 to n do  
    A[k,1] := 1.0*x[k]:  
    A[k,2] := -x[k]*y[k]:  
  end do:  
> p := LinearAlgebra:-LeastSquares(A,y);
```

$$p := \begin{bmatrix} 81.4332038737884574 \\ -0.0107853032506150510 \end{bmatrix}$$

(10)

```
> rplot := plot( p[1]*t/(1+p[2]*t), t = 70..89):
> plots[display]({rplot,pplot});
```



```
> z := Vector([seq( p[1]*x[k]/(1+p[2]*x[k]),k=1..n)]):
> LinearAlgebra:-Norm(z-y,2);
67725.3261260830186
```

(11)

```
>
```

Power model

$$y = a x^b$$