# Answers to short proofs, §3.3, Part II

Note Title 2/24/2009

9 $u, v$ are li. indep, $A$ is nonsingular. Then $Au, Av$ are li. indep

Proof. Let $\alpha Au + \beta Av = 0$

$\therefore \quad A(\alpha u + \beta v) = 0$

$\therefore \quad \alpha u + \beta v = A^{-1} 0 = 0$

Since $u, v$ are li. indep, $\therefore \alpha = \beta = 0$ QED

10 $Au, Av$ are li. dep, $A$ is nonsingular, then $u, v$ are li. dep.

Proof. $\because Au, Av$ are li. dep

$\therefore \exists\ \alpha, \beta$, not both zero, s.t. $\alpha Au + \beta Av = 0$

$\therefore \quad A(\alpha u + \beta v) = 0$ and $\alpha u + \beta v = A^{-1} 0 = 0$ QED

11 $Au, Av$ are li. indep. Then $u, v$ are li. indep

Proof. Let $\alpha u + \beta v = 0$. Then $A(\alpha u + \beta v) = A0 = 0$

$\therefore \quad \alpha(Au) + \beta(Av) = 0$

$\therefore \quad \alpha = \beta = 0$ since $Au, Av$ are li. indep.

12. see classnotes

13 Let $u, v, w$ be li. dep., $x \in \langle u, v, w \rangle$. Then $\exists$ infinitely many $\alpha, \beta, \gamma$ such that $x = \alpha u + \beta v + \gamma w$.

Proof. $\because x \in \langle u, v, w \rangle$

$\therefore \exists\ \alpha_0, \beta_0, \gamma_0$ s.t. $x = \alpha_0 u + \beta_0 v + \gamma_0 w$. not all zero

$\because u, v, w$ are li dep, $\therefore \exists\ \alpha_1, \beta_1, \gamma_1$ s.t. $\alpha_1 u + \beta_1 v + \gamma_1 w = 0$

$\therefore \quad x = x + n0 = (\alpha_0 u + \beta_0 v + \gamma_0 w) + n(\alpha_1 u + \beta_1 v + \gamma_1 w)$

$= (\alpha_0 + n\alpha_1)u + (\beta_0 + n\beta_1)v + (\gamma_0 + n\gamma_1)w$

for $n = 0, 1, 2, \cdots,$ QED

14 If $\alpha u+\beta v+\gamma w = \alpha' u+\beta' v+\gamma' w$ and $\alpha \neq \alpha'$ then $u, v, w$ are li. dep.

Proof. $\because$ $\alpha u+\beta v+\gamma w = \alpha' u+\beta' v+\gamma' w$

$\therefore$ $(\alpha u+\beta v+\gamma w) - (\alpha' u+\beta' v+\gamma' w) = 0$

$\therefore$ $(\alpha-\alpha')u + (\beta-\beta')v + (\gamma-\gamma')w = 0$

$\because$ $\alpha \neq \alpha'$, $\therefore$ $\alpha-\alpha' \neq 0$ $\therefore$ $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ are not all zero. QED.

Answers to proofs for §3.4 Part I.

7 Let $x \in \langle u, v, w \rangle$. If $\exists$ two choices of $\alpha, \beta, \gamma$ s.t. $x = \alpha u+\beta v+\gamma w$. then $(u, v, w)$ is not a basis.

Proof. Let $x = \alpha u+\beta v+\gamma w = \alpha' u+\beta' v+\gamma' w$ and $[\alpha, \beta, \gamma] \neq [\alpha', \beta', \gamma']$. Then

$$0 = x - x = (\alpha u+\beta v+\gamma w) - (\alpha' u+\beta' v+\gamma' w)$$
$$= (\alpha-\alpha')u + (\beta-\beta')v + (\gamma-\gamma')w$$

and $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ are not all zero.

$\therefore$ $(u, v, w)$ are li. dep.

$\therefore$ It is not a basis. QED

8 Let $(u, v, w)$ be a basis for $\mathbb{R}^3$ & $A$ be nonsingular. Then $(Au, Av, Aw)$ is also a basis for $\mathbb{R}^3$.

Proof. We first prove $Au, Av, Aw$ are li. indep

Let $\alpha Au + \beta Av + \gamma Aw = 0$. Then

$A(\alpha u+\beta v+\gamma w) = 0$ $\therefore$ $\alpha u+\beta v+\gamma w = A^{-1}0 = 0$

$\therefore$ $\alpha = \beta = \gamma = 0$ since $(u, v, w)$ is a basis.

We then prove $\mathbb{R}^3 = \langle Au, Av, Aw \rangle$.

$\forall x \in \mathbb{R}^3$. $\exists \alpha, \beta, \gamma$ s.t. $y = A^{-1}x = \alpha u + \beta v + \gamma w$

since $\mathbb{R}^3 = \langle u, v, w \rangle$. and $y \in \mathbb{R}^3$.

$\therefore x = Ay = \alpha(Au) + \beta(Av) + \gamma(Aw)$

$\therefore x \in \langle Au, Av, Aw \rangle$

QED.

10 see classnotes

11 Let $W$ be a subspace of $V$ and $\dim(W) < \dim(V)$

Then $\exists$ infinitely many $x \in V$ s.t. $x \notin W$.

Proof. Let $(w_1, \cdots, w_n)$ be a basis of $W. < V$

$\because \dim(W) < \dim(V)$

$\therefore$ We can expand $(w_1, \cdots, w_n)$ to

$(w_1, \cdots, w_n, w_{n+1}, \cdots, w_m)$ to a basis of $V$

Let $x = w_{n+1}$. Then $x \notin W$ otherwise

$(w_1, \cdots, w_n, x)$ is l.i. indep. in $W$, violating

Thm 3.4.9 (c).

$\therefore x = w_{n+1}, 2w_{n+1}, 3w_{n+1}, \cdots \notin W$. QED.