Note Title 4/9/2009

1. Let (u,v,w) be a basis for V and $L: V \to W$ be linear. If NullSpace(L) = $\langle w \rangle$, prove $(L(u),L(v))$ is lin. Indep.

Proof. Let $\alpha L(u) + \beta L(v) = 0$

$\because$ L is linear

$\therefore$ $\alpha L(u) + \beta L(v) = L(\alpha u + \beta v) = 0$

$\therefore$ $\alpha u + \beta v \in$ NullSpace $(L) = \langle w \rangle$

$\therefore$ $\exists \gamma$ s.t. $\alpha u + \beta v = \gamma w$ or $\alpha u + \beta v - \gamma w = 0$

$\because$ (u, v, w) is a basis and thus li. indep

$\therefore$ $\alpha = \beta = -\gamma = 0$

$\therefore$ $L(u)$ and $L(v)$ are li. indep. QED

2. Let (u,v,w) be a basis for V and $L: V \to W$ be linear. If $(L(u),L(v),L(w))$ is lin. dep., prove dim (NullSpace(L)) > 1.

Proof. $\because$ $(L(u), L(v), L(w))$ is li. dep.

$\therefore$ $\exists\ \alpha, \beta, \gamma$ not all zero s.t.

$\alpha L(u) + \beta L(v) + \gamma L(w) = 0$

or $L(\alpha u + \beta v + \gamma w) = 0$

$\therefore$ $\alpha u + \beta v + \gamma w \in$ NullSpace (L)

$\because$ α, β, γ are not all zero and (u, v, w) is a basis, $\therefore$ $z_1 = \alpha u + \beta v + \gamma w \neq 0$ is li. ind.

$\therefore$ We can expand (z_1) to a basis $(z_1, z_2, \dots, z_n)$ of NullSpace (L)

$\therefore$ dim (NullSpace (L)) $= n \geq 1$ QED

3. Let $V = \langle u,v,w \rangle$ and $L: V \to W$ be linear. If $(L(u),L(v),L(w))$ are li. indep., prove NullSpace(L) = $\{0\}$.

Proof. $\forall z \in \text{NullSpace}(L)$

$\exists\ \alpha, \beta, \gamma$ s.t. $z = \alpha u + \beta v + \gamma w$ since

$\text{NullSpace}(L) \subseteq V = \langle u, v, w \rangle$

$\because\ z \in \text{NullSpace}(L) \quad \therefore\ L(z) = 0$

$\therefore\ \alpha L(u) + \beta L(v) + \gamma L(w) = L(\alpha u + \beta v + \gamma w) = L(z) = 0$

$\because\ (L(u), L(v), L(w))$ are li. indep.

$\therefore\ \alpha = \beta = \gamma = 0$ Namely $z = 0$

$\therefore$ Every vector in NullSpace (L) is a zero vector

$\therefore\ \text{NullSpace}(L) = \{0\}$ QED

Proof 2. $\text{Range}(L) = \langle L(u), L(v), L(w) \rangle$

$\because\ (L(u), L(v), L(w))$ is li. indep

$\therefore$ It is a basis for Range (L)

$\therefore\ \dim(\text{Range}(L)) = 3$

$\because\ \dim(\text{Range}(L)) + \dim(\text{NullSpace}(L))$
$= \dim(V) = 3$

$\therefore\ \dim(\text{NullSpace}(L)) = 0$

$\therefore\ \text{NullSpace}(L) = \{0\}$ QED

4 Prove $P : F_{3,1} \to F_{2,1}$ with

$$P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is linear

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then

$$P(x+y) = P\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Px + Py$$

$$P(\alpha x) = P\begin{bmatrix}\alpha x_1\\ \alpha x_2\\ \alpha x_3\end{bmatrix} = \begin{bmatrix}\alpha x_1\\ \alpha x_2\end{bmatrix} = \alpha\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \alpha P x$$

$\therefore$ P is linear QED.

5. $E : R(z) \to R$ with $Ef = f(0)$

Prove E is linear. what is the range?

Proof $\forall f, g \in R(z)$

$$E(f+g) = (f+g)(0) = f(0) + g(0) = Ef + Eg$$

$$E(\alpha f) = (\alpha f)(0) = \alpha \cdot f(0) = \alpha E(f)$$

$\therefore$ E is linear QED

The range is entire R since $\forall\, r \in R$. Let

$$f(z) = r = r + 0\cdot z + 0\cdot z^2 \cdots$$

$\therefore$ $f \in R(z)$ and $Ef = r$.

6 $T : F_{n,n} \to F_{n,n}$ with $T(A) = P^{-1}AP$ is linear

Proof
$$T(A+B) = P^{-1}(A+B)P = (P^{-1}A + P^{-1}B)P$$
$$= P^{-1}AP + P^{-1}BP = T(A) + T(B)$$
$$T(\alpha A) = P^{-1}(\alpha A)P = \alpha(P^{-1}A)P = \alpha(P^{-1}AP) = \alpha T(A)$$

$\therefore$ T is linear QED