Answers to Eigen-exercises, part I

Note Title 4/9/2009

1. If λ is an eigenvalue of A, then 3λ is an eigenvalue of $3A$

Proof. $\because$ λ is an eigenvalue of A

$\therefore$ $\exists\ x \neq 0$ s.t. $Ax = \lambda x$

$\therefore$ $(3A)x = 3(Ax) = 3(\lambda x) = (3\lambda)x$

$\therefore$ 3λ is an eigenvalue of $3A$ QED

2 Let λ be an eigenvalue of A. Then $\lambda + \delta$ is an eigenvalue of $A + \delta I$

Proof. $\because$ λ is an eigenvalue of A

$\therefore$ $\exists\ x \neq 0$ s.t. $Ax = \lambda x$

$\therefore$ $(A + \delta I)x = Ax + \delta I x = \lambda x + \delta x = (\lambda + \delta)x$

$\therefore$ $(\lambda + \delta)$ is an eigenvalue of $A + \delta I$. QED

3 Let λ be an eigenvalue of A. Then λ^2 is an eigenvalue of A^2

Proof. $\because$ λ is an eigenvalue of A

$\therefore$ $\exists\ x \neq 0$ s.t. $Ax = \lambda x$

$\therefore$ $A^2 x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x)$

$= \lambda^2 x$

$\therefore$ λ^2 is an eigenvalue of A^2 QED

4 Let λ be an eigenvalue of A that is nonsingular. Then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Proof. $\because$ λ is an eigenvalue of A

$\therefore$ $\exists\ x \neq 0$ s.t. $Ax = \lambda x$

$\therefore$ $x = A^{-1}Ax = A^{-1}(\lambda x) = \lambda(A^{-1}x)$

$\because x \neq 0 \quad \therefore \lambda \neq 0$ and $A^{-1}x \neq 0$.

$\therefore A^{-1}x = \frac{1}{\lambda}x$

$\therefore \frac{1}{\lambda}$ is an eigenvalue of A^{-1}

7 Let $x^T x = 1$, $(A - \lambda I)x = b$. Then λ is an eigenvalue of $A - bx^T$

Proof $\because x^T x = 1$, $\therefore x \neq 0$.

$(A - bx^T)x = Ax - (bx^T)x = Ax - b(x^T x)$

$= Ax - b \cdot 1 = Ax - b$

$\because (A - \lambda I)x = b \quad \therefore Ax - \lambda x = b$

$\therefore Ax - b = \lambda x$

$\therefore (A - bx^T)x = \lambda x$ and $x \neq 0$

$\therefore \lambda$ is an eigenvalue of $A - bx^T$ QED

8 Let $x \neq 0$ and $x \in \mathrm{NullSpace}(A)$. Then x is an eigenvector of A

Proof. $\because x \in \mathrm{NullSpace}(A)$

$\therefore Ax = 0 = 0 \cdot x$

$\because x \neq 0$, $\therefore (0, x)$ is an eigenpair of A

QED

9 If $A^2 = I$ and λ is an eigenvalue of A, then $\lambda = \pm 1$

Proof. $\because \lambda$ is an eigenvalue of A

$\therefore \exists x \neq 0$ s.t. $Ax = \lambda x$

$\therefore A^2 x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$

Also, $A^2 x = Ix = x$

$\therefore x = \lambda^2 x$ or $(\lambda^2 - 1)x = 0$ but $x \neq 0$

$\therefore \lambda^2 - 1 = 0$ or $\lambda = \pm 1$ QED