Answer to Homework 04

Note Title 2/10/2009

#1 P108. W is a subspace of V if and only if
$\alpha x + \beta y \in W$ for all $x, y \in W$ and $\alpha, \beta \in F$

Proof. Assume W is a subspace of V. Then W is a vector space. $\therefore$ $\forall\, x, y \in W$, $\alpha x, \beta y \in W$ for all $\alpha, \beta \in F$. $\therefore$ $\alpha x + \beta y \in W$.

Conversely. Assume $\alpha x + \beta y \in W$ for all $x, y \in W$ and $\alpha, \beta \in F$. Then

$$x + y = 1 \cdot x + 1 \cdot y \in W$$
$$\alpha x = \alpha x + 0 \cdot y \in W$$

$\therefore$ W is closed under addition and scalar multiplication

$\therefore$ W is a subspace by Thm 3.2.2. QED

#6(b). Let $W_1 = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$, $W_2 = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$

$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. $u, v \in W_1 \cup W_2$

Then $u + v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin W_1 \cup W_2$

$\therefore$ $W_1 \cup W_2$ is not a subspace. QED

#6(c) Part 1. If W_1 & W_2 are subspaces of V, then $W_1 \cap W_2$ is also a subspace of V.

Proof. For all $x, y \in W_1 \cap W_2$ and $\alpha, \beta \in F$.

$\alpha x + \beta y \in W_1$ since W_1 is a vector space

$\alpha x + \beta y \in W_2$ since W_2 is a vector space.

$\therefore$ $\alpha x + \beta y \in W_1 \cap W_2$

$\therefore$ $W_1 \cap W_2$ is a subspace by problem #1.

#6(c) part 2: If W_1 and W_2 are subspaces of V, so is W_1+W_2.

Proof. $\forall\ x, y \in W_1+W_2$, $\exists\ u_1, v_1 \in W_1$ and $u_2, v_2 \in W_2$ such that $x = u_1 + u_2$ and $y = v_1 + v_2$

$\forall\ \alpha, \beta \in F$. $\alpha x + \beta y = \alpha(u_1+u_2) + \beta(v_1+v_2)$

$= (\alpha u_1 + \beta v_1) + (\alpha u_2 + \beta v_2)$

where $\alpha u_1 + \beta v_1 \in W_1$ and $\alpha u_2 + \beta v_2 \in W_2$ since W_1 and W_2 are subspaces (problem #1)

$\therefore$ $\alpha x + \beta y \in W_1 + W_2$

$\therefore$ W_1+W_2 is a subspace. QED

#6(d) Let W_1, W_2 be subspaces of V and $W_1 \cap W_2 = \{0\}$. Then $\forall\ z \in W_1+W_2$, $\exists$ unique x and y s.t. $z = x+y$.

Proof. Assume $z = x+y = u+v$ for $x, u \in W_1$ and $y, v \in W_2$. Then $0 = z - z = (x+y) - (u+v)$ $= (x-u) + (y-v)$. $\therefore$ $x-u = -(y-v) \in W_2$ and $y-v = -(x-u) \in W_1$. Since $x-u \in W_1$ and $y-v \in W_2$ $\therefore$ $x-u, y-v \in W_1 \cap W_2 = \{0\}$

$\therefore$ $x-u = y-v = 0$

$\therefore$ $x = u$, $y = v$. QED.

#8 p 119. Prove (x_1, x_2, x_3, x_4) is li. dep. when $x_4 = x_1$

Proof. $1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + (-1)x_4 = x_1 - x_4 = 0$ and $1, 0, 0, -1$ are not all zero.

$\therefore$ (x_1, x_2, x_3, x_4) is li. dep. QED.