Answers to HW06

1. Let X = $\left[\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$ be a basis of R^3, Find the representation of the following vectors with respect to X:

(e) $\begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$

Solution (e): Let $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ be the representation. Then

$$\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix} \beta + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \gamma = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

```
> A := <<sqrt(2),sqrt(2),0>|<sqrt(2),-sqrt(2),0>|<0,0,1>>;
```

$$A := \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

```
> b := <0,sqrt(2),0>;
```

$$b := \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix} \tag{2}$$

```
> x := LinearAlgebra:-LinearSolve(A,b);
```

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \tag{3}$$

Therefore, the representation is $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$.

Solution (f). Similarly, let

```
> d := <1,1,sqrt(2)>;
```

$$d := \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} \quad (4)$$

```
> y := LinearAlgebra:-LinearSolve(A,d);
```

$$y := \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix} \quad (5)$$

The representation is $\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$.

Problem 5: Show that $(\tau\text{-}1, \tau^2+\tau+1, \tau^2+1)$ is a basis for $R_3[\tau]$. Then find a representation for
(a) τ^2 (b) τ^2+1

Proof. It suffice to prove that $(\tau\text{-}1, \tau^2+\tau+1, \tau^2+1)$ is linearly independent because we know the dimension of $R_3[\tau]$ is 3.

Let $\alpha(\tau-1)+\beta(\tau^2+\tau+1)+\gamma(\tau^2+1)=0$. Then
$(\beta+\gamma)\cdot\tau^2+(\alpha+\beta)\cdot\tau+(-\alpha+\beta+\gamma)\cdot 1=0$.
Therefore $\beta+\gamma=0$, $\alpha+\beta=0$, $-\alpha+\beta+\gamma=0$.

```
> alpha,beta,gama:='alpha','beta','gama':
solve({beta+gama=0, alpha+beta=0, -alpha+beta+gama=0},{alpha,
beta,gama});
```

$$\{\alpha=0, \beta=0, gama=0\} \quad (6)$$

Therefore, $(\tau\text{-}1, \tau^2 + \tau + 1, \tau^2 + 1)$ is a basis. QED.

Solution (a). To find a representation $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ of τ^2, we have an equation

$\alpha \cdot (\tau - 1) + \beta(\tau^2 + \tau + 1) + \gamma(\tau^2 + 1) = \tau^2$. Namely

$(\beta + \gamma) \cdot \tau^2 + (\alpha + \beta) \cdot \tau + (-\alpha + \beta + \gamma) \cdot 1 = 1 \cdot \tau^2 + 0 \cdot \tau + 0 \cdot 1.$

Therefore, $\beta + \gamma = 1$, $\alpha + \beta = 0$, $-\alpha + \beta + \gamma = 0$

```
> alpha,beta,gama:='alpha','beta','gama':
  solve({beta+gama=1, alpha+beta=0, -alpha+beta+gama=0},{alpha,
  beta,gama});
```

$$\{\alpha = 1, \beta = -1, gama = 2\} \quad \textbf{(7)}$$

Therefore, the representation is $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Similarly for (b):

```
> alpha,beta,gama:='alpha','beta','gama':
  solve({beta+gama=1, alpha+beta=0, -alpha+beta+gama=1},{alpha,
  beta,gama});
```

$$\{\alpha = 0, \beta = 0, gama = 1\} \quad \textbf{(8)}$$

and the representation is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.