Answer to homework 03 (proof problems)

Note Title 2/5/2009

P92, #2. If AB is nonsingular then both A and B are nonsingular

Proof 1. For any $u \neq 0$ $(AB)u \neq 0$ since AB is nonsingular and 2.7.2. $\therefore$ $Bu \neq 0$ since $(AB)x = A(Bx) \neq 0$ $\therefore$ B is nonsingular by 2.7.2. and B^{-1} exists. $\therefore$ $Au = (AB)(B^{-1}u) \neq 0$
$\therefore$ A is non singular. QED

Proof 2. Assume B is singular. Then $Bx=0$ has a nontrivial solution $x=u$. $\therefore$ $(AB)x = A(Bx) = 0$ has a nontrivial solution, which is impossible by 2.7.2 $\therefore$ B is nonsingular and B^{-1} exists.
Assume A is singular. Then $Ax=0$ has a nontrivial solution $x = v \neq 0$. $\therefore$ $Av=0$, $B^{-1}v \neq 0$ and $(AB)(B^{-1}v) = 0$ which is impossible by 2.7.2
$\therefore$ A is nonsingular too QED

P93 #5. An $m \times (m+1)$ system always has a solution.
False. Consider $m=1$

$$0x_1 + 0x_2 = 1$$

P93 #6. Two nonsingular $n \times n$ matrices are row equivalent
True.
Proof. Let A and B be nonsingular $n \times n$ matrices.
Then Thm 2.7.1 implies $A \overset{R}{\sim} I$, $B \overset{R}{\sim} I$
$\therefore$ $A \overset{R}{\sim} I \overset{R}{\sim} B$. QED

P93 #8. Let A be $m \times n$ and rank m. Then $Ax=b$ always has a solution

True

Proof. Thm 2.5.4. QED

P93 #10. If A and B are $n \times n$, AB is nonsingular, then BA is nonsingular

True.

Proof. By Problem #2, p93, both A and B are nonsingular. $\therefore$ A^{-1}, B^{-1} exist

$\therefore$ $(BA)(A^{-1}B^{-1}) = BA^{-1}AB^{-1} = BIB^{-1} = I$ QED.

P94 #1(d) Prove $\alpha 0 = 0$.

Proof. $\because$ $\alpha 0 = \alpha(0+0)$ (A2)

$= \alpha 0 + \alpha 0$ (S1)

$\therefore$ $\alpha 0 + (-\alpha 0) = \alpha 0 + \alpha 0 + (-\alpha 0)$ (A3)

$\therefore$ $0 = \alpha 0$ (A3)

QED

- Prove $\alpha u = 0 \Rightarrow \alpha = 0$ or $u = 0$

Proof. Let $\alpha u = 0$. If $\alpha = 0$ then the statement is true. Assume $\alpha \neq 0$, then $\frac{1}{\alpha}$ exists

$\therefore$ $0 = \frac{1}{\alpha} 0$ (problem 1 d)

$= \frac{1}{\alpha}(\alpha u)$ (substitution)

$= (\frac{1}{\alpha}\alpha) u$ (S3)

$= 1 u = u$ (S4)

Thus either $\alpha = 0$ or $u = 0$ QED

P100 #4 a The set of Polynomials $P(t)$ with complex coefficients satisfying $P(0)=0$ is a vector space.

Proof. Let V be the set of such polynomials

For any $P(t), Q(t) \in V$, we can write

$$P(t) = a_0 + a_1 t + \cdots + a_n t^n, \quad Q(t) = b_0 + b_1 t + \cdots + b_n t^n$$

with $a_0, \cdots, a_n, \ b_0, \cdots, b_n \in C$ $P(0) = Q(0) = 0$

Thus $(P+Q)(t) = (a_0+b_0) + (a_1+b_1)t + \cdots + (a_n+b_n)t^n$

$(\alpha P)(t) = (\alpha a_0) + (\alpha a_1) t + \cdots + (\alpha a_n) t^n$

with $(P+Q)(0) = P(0) + Q(0) = 0 + 0 = 0$

$(\alpha P)(0) = \alpha P(0) = \alpha \cdot 0 = 0$

$\therefore$ V has addition and scalar multiplication

A1. $(P(t) + Q(t)) + R(t) = P(t) + (Q(t) + R(t))$

A2 $0 \in V$ since $P(0)=0$ if $P(t) \equiv 0$.

A3 $-P(t) \in V$ since $-P(0) = -0 = 0$.

A4 $P(t) + Q(t) = Q(t) + P(t)$

S1- S4 are obvious. QED.

P100, 4(b) No. Because $Q(0) = 1$ implies that $0 \notin V$.

P100 4(d) No. Let $x \in V = \{$all real #s$\}$

Then $i \cdot x \notin V$.