Answer to Homework 02 (Proof problems)

Note Title 2/3/2009

1. Prove $(I+A^{-1})^{-1} = A(A+I)^{-1}$

Proof $(I+A^{-1})A(A+I)^{-1} = (A+A^{-1}A)(A+I)^{-1}$

$= (A+I)(A+I)^{-1} = I$

$\therefore (I+A^{-1})^{-1} = A(A+I)^{-1}$ QED

2. Prove $A^{-1}+B^{-1} = A^{-1}(A+B)B^{-1}$

Proof. $A(A^{-1}+B^{-1})B = (AA^{-1}+AB^{-1})B$

$= (I+AB^{-1})B = (B+AB^{-1}B) = B+A = A+B)$

$\therefore A^{-1}+B^{-1} = A^{-1}[A(A^{-1}+B^{-1})B]B^{-1} = A^{-1}(A+B)B^{-1}$ QED

3. Prove: If $B^TB = I$, then $(BB^T)^2 = I$

Proof. $(BB^T)^2 = (BB^T)(BB^T) = B(B^TB)B^T = BIB^T = BB^T$ QED

4. Prove: If A and B are invertible, so are AB^{-1} and BA^{-1}

Proof. $\because$ A and B are invertible

$\therefore$ A^{-1} and B^{-1} exist

$\therefore (AB^{-1})(BA^{-1}) = A(B^{-1}B)A^{-1} = AIA^{-1} = I$

$(BA^{-1})(AB^{-1}) = B(A^{-1}A)B^{-1} = BIB^{-1} = I$

$\therefore$ Both AB^{-1} and BA^{-1} are invertible QED

5. Prove $(I+A)^{-1}$ and A commute (i.e. $(I+A)^{-1}A = A(I+A)^{-1}$)

Proof. $\because$ $A(I+A) = A+AA = (I+A)A$

$\therefore (I+A)^{-1}[A(I+A)](I+A)^{-1} = (I+A)^{-1}[(I+A)A](I+A)^{-1}$

Namely $(I+A)^{-1}A = A(I+A)^{-1}$ QED

Alternative proof. $(I+A)^{-1}A = (AA^{-1}+A)^{-1}A = [A(A^{-1}+I)]^{-1}A$

$= (A^{-1}+I)^{-1}A^{-1}A = (A^{-1}+I)^{-1} = (A^{-1}+AA^{-1})^{-1} = [(I+A)A^{-1}]^{-1} = A(I+A)^{-1}$

QED

Extra credit problem 1. Prove $(I+AB)^{-1} = I - A(I+BA)^{-1}B$

Proof. $B(I+AB) = B + BAB = (I+BA)B$

$\therefore (I+BA)^{-1}[B(I+AB)](I+AB)^{-1} = (I+BA)^{-1}[(I+BA)B](I+AB)^{-1}$

$\therefore (I+BA)^{-1}B = B(I+AB)^{-1}$

$\therefore I - A(I+BA)^{-1}B = I - AB(I+AB)^{-1}$

$= (I+AB)(I+AB)^{-1} - AB(I+AB)^{-1}$

$= (I+AB-AB)(I+AB)^{-1}$

$= I(I+AB)^{-1} = (I+AB)^{-1}$ QED

Extra credit problem #2

Prove: $A - A(A+B)^{-1}A = B - B(A+B)^{-1}B$

Proof. $LHS = A[I - (A+B)^{-1}A]$

$= A[(A+B)^{-1}(A+B) - (A+B)^{-1}A]$

$= A(A+B)^{-1}[\cancel{A}+B-\cancel{A}]$

$= A(A+B)^{-1}B = (A+B-B)(A+B)^{-1}B$

$= (A+B)(A+B)^{-1}B - B(A+B)^{-1}B = RHS$

QED