Math 343 Exam 2

April 28, 2009

Name (print) ____________________ Student # ____________________

Signature ____________________ Dr. Z. Zeng

1. (15 point) Consider the linear transformation L from $R_4[\tau]$ to $R_3[\tau]$ with bases $X = (1, \tau, \tau^2, \tau^3)$ and $Y = (1, \tau, \tau^2)$ respectively, where $L(p(\tau)) = \tau p''(\tau) + p(0)$ for any $p(\tau) \in R_4[\tau]$. Find the matrix representation for this linear transformation.

$$L(1) = \tau \cdot 0 + 1 = 1 = [1, \tau, \tau^2] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$L(\tau) = \tau \cdot 0 + 0 = 0 = [1, \tau, \tau^2] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L(\tau^2) = \tau \cdot 2 + 0 = 2\tau = [1, \tau, \tau^2] \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$L(\tau^3) = \tau \cdot (6\tau) + 0 = 6\tau^2$$

$$= [1, \tau, \tau^2] \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

The matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

2. Let λ be an eigenvalue of the matrix A. Prove that $5\lambda - 3$ is an eigenvalue of $5A - 3I$.

Proof. $\because$ λ is an eigenvalue of A

$\therefore$ $\exists\ x \neq 0$ s.t. $Ax = \lambda x$

$(5A - 3I)x = 5(Ax) - 3x = 5(\lambda x) - 3x = (5\lambda - 3)x$

$\therefore$ $5\lambda - 3$ is an eigenvalue of $5A - 3I$.

QED

3. (15 points) Let A be a nonsingular matrix and λ is an eigenvalue of A. Prove that $\lambda \neq 0$.

Proof. $\because$ λ is an eigenvalue of A

$\therefore$ $\exists\ x \neq 0$ s.t. $Ax = \lambda x$

$\because$ A is nonsingular

$\therefore$ A^{-1} exists

$\therefore$ $x = A^{-1}Ax = A^{-1}(\lambda x) = \lambda(A^{-1}x)$

$\therefore$ $\lambda \neq 0$ otherwise $x=0$ would be a contradiction QED.

4. (15 points) Find the Householder matrix H such that

$$H\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$u = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \|u\|_2 = \sqrt{(-1)^2+1^2+2^2} = \sqrt{6}$$

normalize u:

$$u = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$H = I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{\sqrt{6}^2}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}[-1, 1, 2]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

5. (25 points) Given the following singular value decomposition of A

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^T$$

Answer the following questions (you may continue to the next page)

(a) What is the rank of A?

(b) What are the solutions of $Ax = 0$?

(c) Find a basis for the range of A.

(d) Find the rank-1 approximation of A.

(e) Write down a non-fullspan SVD.

(f) If we store only the rank-1 approximation of A, how many percent of storage would be saved?

(a) $\operatorname{rank}(A) = 2$

(b) $x \in \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ or $x = s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$

(d) $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} [2] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} [0\ 0\ 2\ 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}^T$

$\therefore$ saved 11 numbers

or saved $\frac{11}{20} = 55\%$

(f) Original A is $5 \times 4 = 20$

rank-1 approx. needs 9 numbers

6. (15 points) Let the QR decomposition matrix A be given as follows, along with the vector b:

$$A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

Find the least squares solution to the linear system $Ax = b$.

(1) $Q = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

(2) $d = Q^T b = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} = \begin{bmatrix} 18 \\ -12 \end{bmatrix}$

(3) solve $Rx = d$

$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18 \\ -12 \end{bmatrix}$,

Solution

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}$

$3x_2 = -12$ or $x_2 = -4$

$2x_1 + x_2 = 18$, $2x_1 - 4 = 18$ $\quad x_1 = \frac{22}{2} = 11$

7. (Extra 15 points) The so-called *Shifted QR Algorithm* repeats the following process

- QR decomposition $\quad A - \sigma I = QR$
- reverse multiplication $\quad B = RQ + \sigma I$

Prove that $B = Q^T AQ$.

Proof $\because A - \sigma I = QR$, $\therefore Q^T(A - \sigma I) = Q^T QR = R$

$\therefore B = RQ = Q^T(A - \sigma I)Q + \sigma I$

$= Q^T AQ - Q^T \sigma I Q + \sigma I$

$= Q^T AQ - \sigma Q^T Q + \sigma I$

$= Q^T AQ - \sigma I + \sigma I = Q^T AQ$

QED