Review for Exam 2

Note Title 4/23/2009

1. Matrix representation of a linear transformation $L : V \longrightarrow W$ with bases X, Y or V and W respectively

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d] & W \arrow[d] \\ \vec{a} = R_X v \arrow[r, "A"] & \vec{b} = A\vec{a} \end{tikzcd}$$

Possible questions : $A = ?$

§ 5.2. Exercises on p 229-231

For Example $V = R_1[\tau]$, $W = R_2[\tau]$

bases $\{1+\tau, 2\tau\}$ $\{1, \tau, \tau^2\}$

$6 + 5\tau \in V$

$\hookrightarrow 6(1+\tau) - \frac{1}{2}(2\tau)$: $\begin{bmatrix} 6 \\ -\frac{1}{2} \end{bmatrix}$

$L(p) = \tau \cdot p + p'$

$L(1+\tau) = \tau(1+\tau) + (1+\tau)'$

$= \tau + \tau^2 + 1$ $\sim \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$L(2t) = t(2t) + (2t)'$$
$$= 2t^2 + 2 \quad \sim \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

If $$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \sim G$$

$$G(2t) = ?$$

$$\begin{tikzcd} 2t \arrow[r, "G"] \arrow[d] & ? \arrow[d] \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \arrow[r] & B\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{tikzcd}$$

$$? = 0 \cdot 1 + 1 \cdot t + 0 \cdot t^2 = t$$

$$B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

2. Eigen proofs

Study the eigen-exercises

Key : To prove λ is an

eigenvalue of A: Argue $\exists\ x \neq 0$

s.t. $Ax = \lambda x$

3. Householder transformation

$$H : x \longrightarrow y = Hx \quad \text{if} \quad \|x\|_2 = \|y\|_2$$

① $u = x - y$

② $v = u / \|u\|_2$

③ $H = I - 2vv^T$

Example: what is H s.t.

$$H = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

① $u = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$

② $v = \frac{1}{\sqrt{20}} \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$

③ $H = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{20} \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} [-2, 0, 4]$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} 4 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & 16 \end{bmatrix} = -$$

4. Given transformation

5. QR decomposition

$$A = \begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} \qquad Q^T Q = Q Q^T = I$$

(full span)

$$= \begin{bmatrix} \\ Q_1 \\ \\ \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix} \qquad Q_1^T Q_1 = I$$

Applications

1. Solve $Ax = b$ (regular or over determined)

① $A = Q_1 R_1$ (non full span)

② $d = Q^T b$

③ $R_1 x = d$ (backward substitution)

Skip solve $Ax = 0$ by QR

(Least-Square solution to $Ax = b$

$\Leftrightarrow A^T A x = A^T b$)

6. SVD

(a) rank

(b) 4 fundamental subspaces

(c) Solve $Ax=0$ (NullSpace(A))

(d) Solve Least-squares solution to $Ax=b$ (overdetermined)

(e) matrix approximation

$$m\underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{n} =: \underbrace{\begin{bmatrix} \underbrace{U_1}_{r} & U_2 \end{bmatrix}}_{m \times m} \underbrace{\begin{bmatrix} \overbrace{\sigma_1 \; \ddots \; \sigma_r}^{r} & & \\ & 0 & \\ & & \ddots \; 0 \end{bmatrix}}_{m \times n} \begin{bmatrix} \underbrace{V_1}_{r} & V_2 \end{bmatrix}^T$$

$Ax=0$

$\underbrace{\text{Columns of}}$

$U_1 \rightarrow$ Range (A)

$V_2 \rightarrow$ NullSpace (A)

$U_2 \rightarrow$ Left NullSpace (A)

$V_1 \rightarrow$ RowSpace (A)

Say a rank-3 Appromation

$$A \approx \begin{bmatrix} \vdots & \vdots & \vdots \\ u_1 & u_2 & u_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ v_1 & v_2 & v_2 \\ \vdots & \vdots & \vdots \end{bmatrix}^T$$

$$U_1 \qquad\qquad \Sigma_1 \qquad\qquad V_1^T$$