Note Title 4/21/2009

1. How to solve $Ax = b$ if we have $A = U \Sigma V^T$ (SVD)

$$Ax = b \iff U \Sigma V^T x = b$$

$$U y = b \quad (y = \Sigma V^T x)$$

$$y = U^T b$$

$$\Sigma V^T x = y \qquad \Sigma^{-1} = \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix}$$

$$z = V^T x = \Sigma^{-1} y$$

$$V^T x = z \qquad x = V z$$

$$\boxed{\begin{array}{ll} 1. & y = U^T b \\ 2. & z = \Sigma^{-1} y \\ 3. & x = V z \end{array}}$$

$$U \Sigma V^T x = b$$

$$\Sigma V^T x = U^T b = y$$

$$V^T x = \Sigma^{-1} y = z$$

$$x = U z$$

2. $A = U \Sigma V^T$

Proof 1 $A^{-1} = (V^T)^{-1} \Sigma^{-1} U^{-1}$

$$= V \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix} U^T$$

Proof 2 $A = U \Sigma V^T$

$$A (V \cdot \Sigma^{-1} U^T) = (U \Sigma V^T)(V \Sigma^{-1} U^T)$$

$$= U \Sigma \cancel{(V^T V)} \Sigma^{-1} U^T$$

$$= U \cancel{\Sigma \Sigma^{-1}} U^T$$

$$= U U^T = I$$

Continue : Application 4

solve $Ax = b$ for LS solution

or $A^T A x = A^T b$ (A is tall)

use non-fullspan version

$$A = U_1 \Sigma_1 V_1^T$$

$U_1^T U_1 = I$ but $U_1 U_1^T \neq I$

$V_1^T V_1 = I$ but $V_1 U_1^T \neq I$

$$\Sigma_1^{-1} = \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_r \end{pmatrix} \qquad \Sigma_1^T = \Sigma$$

$$A^T A = V_1 \Sigma_1 \cancel{U_1^T U_1} \Sigma_1 V_1^T$$

$A^T A x = A^T b$ $\qquad V_1 \Sigma_1 \Sigma_1 V_1^T x = V_1 \Sigma_1 U_1^T b$

$$\Sigma_1 \Sigma_1 V_1^T x = \Sigma_1 U_1^T b$$

$$\Sigma_1 V_1^T x = U_1^T b \quad = y$$

$$V_1^T x = \Sigma_1^{-1} U_1^T b \quad = z$$

$$x = V_1 z$$

Algorithm

1. $A = U_1 \Sigma_1 V_1^T$
2. $y = U^T b$
3. $z = \Sigma_1^{-1} y$
4. $x = V_1 z$

Consider fullspan SVD

$$A = [U_1, U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} [V_1, V_2]^T$$

$$= U_1 \Sigma_1 V_1^T$$

$$[V_1, V_2]^T [V_1, V_2] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} [V_1, V_2] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} V_1^T V_1 & V_1^T V_2 \\ V_2^T V_1 & V_2^T V_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$V_1^T V_2 = 0, \quad V_2^T V_1 = 0$$

Revisit $V_1^T x = z$

$$V_1^T(V_1 x + V_2 u) = z + 0$$

The solution to $V_1^T x = z$

$$x_0 + V_2 u \qquad \forall u$$

($x_0 = V_1 z$ is the minimum norm solution)

Application 5

Image processing & approximation

1. Image file $\to$ matrix A (Image2Matrix.mpl)

2. $A = [\underbrace{U_1}_{r}, U_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} [\underbrace{V_1}_{r}, V_2]^T$

$\approx U_1 \Sigma_1 V_1^T = B$

3. matrix $B \to$ image array Q (Matrix2Image.mpl)

4. ImageTools:-Preview (Q)

need much less storage

Ex

Exam 2

1. Proof problems

2. Ask some "how to" question on computation

Final exam (optional)