Summary of QR decomposition

Note Title 4/14/2009

1. Every matrix $A = QR$

   $Q^T Q = I$, R is upper-triangular

2. Fullspan QR

$$\underbrace{\begin{bmatrix} \; A \; \end{bmatrix}}_{m \times n} = \underbrace{\begin{bmatrix} \; Q \; \end{bmatrix}}_{m \times m} \underbrace{\begin{bmatrix} \; R \; \end{bmatrix}}_{m \times n}$$

3. Non-fullspan

$$\underbrace{\begin{bmatrix} \; A \; \end{bmatrix}}_{m \times n} = \underbrace{\begin{bmatrix} \; Q \; \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} R \end{bmatrix}}_{n \times n}$$

4. To solve $Ax = b \neq 0$ (regular or overdetermined)

   (1) $A = QR$ (non-fullspan)

   (2) $d = Q^T b$

   (3) solve $Rx = d$

   (The sol. is regular or LS)

5. To solve $Ax = 0$

   (1) $A = QR$ (non-fullspan)

   (2) solve $R^T z = y$, for random y

   (3) Solve $Ru = z$

   (4) $x = u / \|u\|_2$

Singular value decomposition (SVD)

Every matrix A can be written as $A = U\Sigma V^T$

$$\underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{m\times n} = \underbrace{\begin{bmatrix} & & & \\ & & U & \\ & & & \end{bmatrix}}_{m\times m} \underbrace{\overset{\Sigma}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & \\ & & \end{bmatrix}}}_{m\times n} \underbrace{\begin{bmatrix} V^T \end{bmatrix}}_{n\times n}$$

$U^T U = I$, $V^T V = I$ (orthogonal matrices)

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n$ — singular values

* If $\operatorname{rank}(A) = r < n$ (A is rank-deficient)

$\sigma_1 \geq \cdots \geq \sigma_r > 0$ $\quad \sigma_{r+1} = \cdots = \sigma_n = 0$

1st application of SVD : rank-revealing

Application 2 : Solve $Ax = 0$

(1) $A = U\Sigma V^T$ $\quad$ write $V = [v_1, v_2, \cdots, v_n]$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $\quad \sigma_{r+1} \approx \cdots \approx \sigma_n \approx 0$

$$\text{NullSpace}(A) = \langle \underbrace{v_{r+1}, v_{r+2}, \cdots, v_n}_{\text{basis}} \rangle$$

Application 3. The 4 fundamental subspaces

Range(A), NullSpace(A)

RowSpace(A), LeftNullSpace(A)

(Range(A) is also called the column space of A)

$\text{LeftNullSpace}(A) = \{ x \mid x^T A = 0 \}$

To extract bases for The 4 subspaces

$$A = \left[\underbrace{\overbrace{U_1}^{r}}_{\text{Range}(A)} \;\; \underbrace{U_2}_{\text{LeftNullSpace}(A)} \right] \begin{bmatrix} \overbrace{\sigma_1 \cdots \sigma_r}^{r} & \\ & 0 \cdots 0 \end{bmatrix} \left[\underbrace{\overbrace{V_1}^{r}}_{\text{RowSpace}(A)} \;\; \underbrace{V_2}_{\text{NullSpace}(A)} \right]^T$$

$$A = U_1 \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} V_1^T$$ non-fullspan SVD

Application 4 solving LS problem $Ax=b$ that is over determined

Thm: x is the solution of The normal eq

$$A^TA\,x = A^T b$$

substitution $\qquad A^T = V_1 S_1^T U_1^T = V_1 S_1 U_1^T$

$$A^TA = V_1 S_1 \cancel{U_1^T U} S_1 V_1^T = V_1 S_1^2 V_1^T$$

$$A^Tb = V_1 S_1 U_1^T b$$

$$A^TA\,x = A^Tb \iff V_1 S_1^2 V_1^T x = V_1 S_1 U_1^T b$$

$$S_1^{-1} \cdot \cancel{V_1^T V_1} S_1^2 V_1^T x = \cancel{V_1^T V_1} S_1 U_1^T b$$

$$S_1 V_1^T x = U_1^T b$$

Algorithm : To solve the over determined system $Ax=b$ for LS solution.

(1) $A = U_1 \Sigma_1 V_1^T$ (non full span version)

(2) $d = U_1^T b$ (New RHS)

(3) solve $\Sigma_1 y = d$ (unfinished)

HW. 1. Given $A = U \Sigma V^T$, how to solve the regular system $Ax=b$ (not $A^T A x = A^T b$)

2. Given $A = U \Sigma V^T$ $(n \times n)$

Prove

$$A^{-1} = U \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix} V^T$$