Introduction to eigenvalues

Note Title 4/2/2009

Let $A \in F_{n \times n}$ (square $n \times n$)

Definition: If $\lambda \in \mathbb{C}$ $x \in \mathbb{C}_{n,1}$

(*) $x \neq 0$ s.t.

$$Ax = \lambda x$$

Then λ : an eigenvalue of A

x : an eigenvector of A associated with λ

or (λ, x) is an eigenpair of A.

Example: Prove: If (λ, x) is an e.p. of A, so is $(\lambda, \sigma x)$ for any $\sigma \neq 0$.

Proof. $\because$ (λ, x) is an e.p. of A

$\therefore$ $Ax = \lambda x$, $x \neq 0$

$\therefore$ $A(\sigma x) = \sigma(Ax) = \sigma(\lambda x) = \lambda(\sigma x)$

and $\sigma x \neq 0$ since $\sigma \neq 0$, $x \neq 0$

$\therefore$ $(\lambda, \sigma x)$ is also an eigenpair of A

QED.

EigenExercise #t

λ is an e.v. of A $\Rightarrow$ λ is an e.v. of BAB^{-1}

Proof $\because$ λ is an e.v. of A

$\therefore$ $\exists\ x \neq 0$ s.t. $Ax = \lambda x$

$\therefore$ $A B^{-1} B x = \lambda B^{-1} B x$

$(BAB^{-1}) Bx = \lambda \cancel{BB^{-1}} Bx$ or $(BAB^{-1}) y = \lambda y$

for $y = Bx \neq 0$ (since otherwise

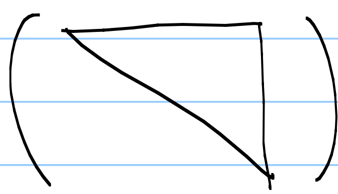
$Bx = 0, \quad x = B^{-1}0 = 0$ contradiction)

$\therefore (\lambda, Bx)$ is an e.p. of BAB^{-1} QED.

If $B^{-1} = B^T$ then B is called an orthogonal matrix.

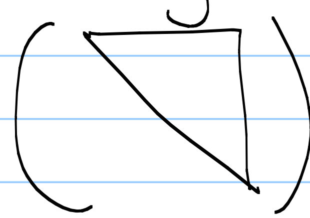
(*) We can prove A, QAQ^T have the same set of eigenvalues if Q is orthogonal.

(*) If a matrix is upper-triangular

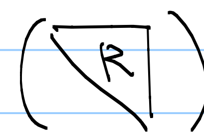


, then the diagonal entries are eigenvalues

The idea of finding eigenvalues:
Can we find orthogonal matrix Q, s.t.

$Q^T A Q =$ 

Start from matrix A

$A = QR$ or $Q^T A =$ 

$$\begin{aligned} A = A_1 &= Q_1 R_1 \\ A_2 &= R_1 Q_1 \\ &= Q_2 R_2 \\ A_3 &= R_2 Q_2 \end{aligned}$$

(prove A_1, A_2 have the same eigenvalue)

Thm $A = QR$ and $B = RQ$
have same eigenvalues if $Q^T Q = I$

Proof $\because$ $A = QR$

$\therefore$ $Q^T A = Q^T Q R = R$

$\therefore$ $B = RQ = (Q^T A) Q = Q^T A Q$

$\therefore$ $B = Q^T A Q$ have the same eigenvalues ~~of~~ as A.

Practice : If $P^2 = P$ then eigenvalues of P are either 0 or 1

Proof. Let (λ, x) be an eigenpair of P

$\therefore$ $Px = \lambda x$

$\therefore$ $P(Px) = P(\lambda x) = \lambda Px$

$P^2 x = Px$

$\therefore$ $Px = \lambda Px$

$\lambda x = \lambda \lambda x$

$\lambda x - \lambda^2 x = 0$ $\quad \lambda(\lambda - 1) x = 0$

$\because$ $x \neq 0$ $\quad \therefore$ $\lambda(\lambda - 1) = 0$ $\quad \therefore$ $\lambda = 0, 1$ QED.

HW : EigenExercise
choose 4 from #1, 2, 3, 4, 7, 8, 9
extra credit for doing more.