Google: QR decomposition Householder, you can get the presentation on QR decomposition:

http://www.zib.de/Optimization/Project-Quito/cursos/curso_2005_2/qr.pdf

Givens Rotation (again, google it)

```
> GivensMatrix := proc(  v :: Vector  # a 2-dim vector
                      )
     local c, s, r, G;

     r := sqrt( v[1]^2 + v[2]^2 );
     c, s := v[1]/r, v[2]/r;

     return <<c,-s>|<s,c>>;
  end proc:
> ClearMatrix := proc(    A :: Matrix,
                        tol :: numeric   )
     local m, n, i, j;
     m, n := LinearAlgebra:-Dimension(A);
     for i from 1 to m do
        for j from 1 to n do
           if abs(A[i,j]) < tol then
              A[i,j] := 0
           end if
        end do
     end do;
     return A;
  end proc:
> v := <a,b>;
```

$$v := \begin{bmatrix} a \\ b \end{bmatrix} \quad (1)$$

```
> G := GivensMatrix(v);
```

$$G := \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix} \quad (2)$$

```
> simplify(G.v);
```

(3)

$$\begin{bmatrix} \sqrt{a^2+b^2} \\ 0 \end{bmatrix} \tag{3}$$

We can use Givens' rotation for QR decomposition on "almost triangular"

```
> A := LinearAlgebra:-RandomMatrix(5,5,generator=-9..9,
  outputoptions=[shape=Hessenberg]);
```

$$A := \begin{bmatrix} -5 & 4 & -3 & 1 & 1 \\ -5 & 6 & 1 & -6 & -2 \\ 0 & -8 & 1 & 1 & -4 \\ 0 & 0 & -2 & 6 & 1 \\ 0 & 0 & 0 & -1 & -7 \end{bmatrix} \tag{4}$$

```
> A := 1.0*A;
```

$$A := \begin{bmatrix} -5. & 4. & -3. & 1. & 1. \\ -5. & 6. & 1. & -6. & -2. \\ 0. & -8. & 1. & 1. & -4. \\ 0. & 0. & -2. & 6. & 1. \\ 0. & 0. & 0. & -1. & -7. \end{bmatrix} \tag{5}$$

```
> G1 := LinearAlgebra:-IdentityMatrix(5)[1..-1,1..-1]:
  T := simplify(GivensMatrix(A[1..2,1])):
  G1[1..2,1..2] := T:
  G1;
```

$$\begin{bmatrix} -0.7071067812 & -0.7071067812 & 0 & 0 & 0 \\ 0.7071067812 & -0.7071067812 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

```
> UseHardwareFloats:=false:
  Digits := 6:
```

```
> A2 := G1.A;
```

$$A2 := \begin{bmatrix} 9.21955 & -1.95237 & -1.95237 & -0.32539 \\ 0. & 6. & 0. & 6. \\ 0. & 0. & 1. & 1. \\ 0. & 2.27777 & 2.27777 & 4.98940 \end{bmatrix} \tag{7}$$

```
> G2 := LinearAlgebra:-IdentityMatrix(5)[1..-1,1..-1]:
  T := simplify(GivensMatrix(A2[2..3,2])):
  G2[2..3,2..3] := T:
  G2;
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -0.1740776559 & -0.9847319278 & 0 & 0 \\ 0 & 0.9847319278 & -0.1740776559 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(8)}$$

```
> A3 := G2.G1.A;
```

$A3$:= [[7.07106781200000078, -7.07106781200000078, 1.41421356240000007, 3.53553390600000039, 0.707106781200000034], [0., 8.12403840428457969, -0.492365964030839210, -1.84637236439603126, 3.56965323837312942], [0., $-4.23057588960773501\ 10^{-10}$, -2.95932015114611512, 4.70009671078070124, 2.78524249503458598], [0., 0., -2., 6., 1.], [0., 0., 0., -1., -7.]] **(9)**

```
> G3 := LinearAlgebra:-IdentityMatrix(5)[1..-1,1..-1]:
  T := simplify(GivensMatrix(A3[3..4,3])):
  G3[3..4,3..4] := T:
  G3;
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.8285293039 & -0.5599457049 & 0 \\ 0 & 0 & 0.5599457049 & -0.8285293039 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(10)}$$

```
> A4 := G3.G2.G1.A;
```

$A4$:= [[7.07106781200000078, -7.07106781200000078, 1.41421356240000007, 3.53553390600000039, 0.707106781200000034], [0., 8.12403840428457969, -0.492365964030839210, -1.84637236439603126, 3.56965323837312942], [0., $3.50515616531765772\ 10^{-10}$, 3.57177487464633358, -7.25384208544581456, -2.86760073050370500], [0., $-2.36889396987294276\ 10^{-10}$, $-2.58285934306989874\ 10^{-10}$, **(11)**

-2.33937685758372859, 0.731055268299576122],
[0., 0., 0., -1., -7.]]

```
> G4 := LinearAlgebra:-IdentityMatrix(5)[1..-1,1..-1]:
  T := simplify(GivensMatrix(A4[4..5,4])):
  G4[4..5,4..5] := T:
  G4;
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.9195132337 & -0.3930590450 \\ 0 & 0 & 0 & 0.3930590450 & -0.9195132337 \end{bmatrix} \quad (12)$$

```
> R := ClearMatrix(G4.G3.G2.G1.A, 10.0^(-8));
```

R := [[7.07106781200000078, -7.07106781200000078, 1.41421356240000007, 3.53553390600000039, 0.707106781200000034], **(13)**
[0., 8.12403840428457969, -0.492365964030839210, -1.84637236439603126, 3.56965323837312942],
[0., 0., 3.57177487464633358, -7.25384208544581456, -2.86760073050370500],
[0., 0., 0., 2.54414702415975924, 2.07919832123243564],
[0., 0., 0., 0., 6.72394052150005006]]

```
> Q := LinearAlgebra:-Transpose( G4.G3.G2.G1 );
```

Q := [[-0.707106781200000034, -0.123091490942290188, -0.576913756444740744, -0.358514663330193716, 0.153252205648013673], **(14)**
[-0.707106781200000034, 0.123091490942290188, 0.576913756444740744, 0.358514663330193716, -0.153252205648013673],
[0., -0.984731927799999962, 0.144228439067370734, 0.0896286658053205704, -0.0383130514003644785],
[0., 0., -0.559945704899999952, 0.761843659444299104, -0.325660936945448788],
[0., 0., 0., -0.393059044999999996, -0.919513233700000042]]

```
> ClearMatrix(A-Q.R,10.0^(-8));
```

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. \end{bmatrix} \quad (15)$$

```
>
```

Example 2: An arrowhead matrix

```
> A := LinearAlgebra:-RandomMatrix(4,4,generator=-9..9):
  A[2,1],A[3,1],A[3,2],A[1,2],A[1,3],A[2,3] := 0,0,0,0,0,0:
  A := 1.0*A;
```

$$A := \begin{bmatrix} 1. & 0. & 0. & 4. \\ 0. & -9. & 0. & 5. \\ 0. & 0. & 1. & -6. \\ 5. & -5. & 8. & -6. \end{bmatrix} \quad \textbf{(16)}$$

```
> G1 := LinearAlgebra:-IdentityMatrix(4)[1..-1,1..-1]:
  T := GivensMatrix( A[[1,4],1] ):
  G1[[1,4],[1,4]] := T:
  G1;
```

$$\begin{bmatrix} 0.196116 & 0 & 0 & 0.980581 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.980581 & 0 & 0 & 0.196116 \end{bmatrix} \quad \textbf{(17)}$$

```
> tol := 10.0^(-4):
  A2 := ClearMatrix(G1.A,tol);
```

$$A2 := \begin{bmatrix} 5.09902 & -4.90290 & 7.84465 & -5.09903 \\ 0. & -9. & 0. & 5. \\ 0. & 0. & 1. & -6. \\ 0. & -0.980580 & 1.56893 & -5.09902 \end{bmatrix} \quad \textbf{(18)}$$

```
> G2 := LinearAlgebra:-IdentityMatrix(4)[1..-1,1..-1]:
  T := GivensMatrix( A2[[2,4],2] ):
  G2[[2,4],[2,4]] := T:
  G2;
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.994117 & 0 & -0.108312 \\ 0 & 0 & 1 & 0 \\ 0 & 0.108312 & 0 & -0.994117 \end{bmatrix} \quad \textbf{(19)}$$

```
> A3 := ClearMatrix(G2.G1.A,tol);
```

$$A3 := \begin{bmatrix} 5.09902 & -4.90290 & 7.84465 & -5.09903 \\ 0. & 9.05326 & -0.169934 & -4.41829 \\ 0. & 0. & 1. & -6. \\ 0. & 0. & -1.55970 & 5.61058 \end{bmatrix} \quad \textbf{(20)}$$

```
> G3 := LinearAlgebra:-IdentityMatrix(4)[1..-1,1..-1]:
```

```
T := GivensMatrix( A3[[3,4],3] ):
G3[[3,4],[3,4]] := T:
G3;
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.539741 & -0.841834 \\ 0 & 0 & 0.841834 & 0.539741 \end{bmatrix} \tag{21}$$

```
> R := ClearMatrix(G3.G2.G1.A,tol);
```

$$R := \begin{bmatrix} 5.09902 & -4.90290 & 7.84465 & -5.09903 \\ 0. & 9.05326 & -0.169934 & -4.41829 \\ 0. & 0. & 1.85275 & -7.96163 \\ 0. & 0. & 0. & -2.02275 \end{bmatrix} \tag{22}$$

```
> Q := LinearAlgebra:-Transpose( G3.G2.G1 );
```

$$Q := \begin{bmatrix} 0.196116 & 0.106209 & -0.820630 & 0.526146 \\ 0. & -0.994117 & -0.0911807 & 0.0584604 \\ 0. & 0. & 0.539741 & 0.841834 \\ 0.980581 & -0.0212417 & 0.164126 & -0.105229 \end{bmatrix} \tag{23}$$

```
> A-Q.R;
```

$$\begin{bmatrix} 0.000001 & -0.000001 & 0.00001 & -0.00003 \\ 0. & 0. & 0.000001 & 0. \\ 0. & 0. & -0.00001 & 0.00004 \\ 0. & 0. & 0. & 0.00002 \end{bmatrix} \tag{24}$$

```
>
```

Application of QR decomposition: Solve Ax = b

```
> # create a linear system
A := 1.0*LinearAlgebra:-RandomMatrix(5,5,generator=-9..9):
b := A.<1,2,3,4,5>:
A, b;
```

$$\begin{bmatrix} -2. & 2. & -2. & -5. & 5. \\ 1. & 1. & -2. & 9. & 6. \\ -1. & 7. & 0. & 8. & 6. \\ -3. & 8. & 6. & 7. & -7. \\ -8. & 2. & -2. & -2. & -6. \end{bmatrix}, \begin{bmatrix} 1. \\ 63. \\ 75. \\ 24. \\ -48. \end{bmatrix} \tag{25}$$

To solve $Ax = b$

1. $A = QR$ $\quad (QRx = b)$

2. $Q^T Q R x = Q^T b \Rightarrow Rx = Q^T b$

3. back substitution on $Rx = b$ to get x

```
> # Step 1:  get the QR decomposition of A
  (Q,R) := LinearAlgebra:-QRDecomposition(A,fullspan);
```

$$Q, R := \begin{bmatrix} -0.22502 & -0.0772387 & -0.408021 & -0.693334 & -0.544253 \\ 0.112509 & -0.171786 & -0.474820 & 0.675006 & -0.526061 \\ -0.112509 & -0.669829 & -0.473232 & -0.0490755 & 0.558864 \\ -0.337527 & -0.641862 & 0.606888 & 0.072749 & -0.317008 \\ -0.900071 & 0.322263 & -0.125776 & 0.236562 & 0.119326 \end{bmatrix}, \begin{bmatrix} 8.88818 & -5.62546 & 0. & 0.67505 & 6.63802 \\ 0. & -9.50554 & -3.99765 & -11.6560 & -2.87642 \\ 0. & 0. & 5.65850 & -1.51936 & -11.2220 \\ 0. & 0. & 0. & 9.18513 & -1.63967 \\ 0. & 0. & 0. & 0. & -1.02140 \end{bmatrix} \tag{26}$$

```
> # Step 2:  Multiply by Q transpose
```

```
> d := LinearAlgebra:-Transpose(Q).b;
```

$$d := \begin{bmatrix} 33.5276 \\ -92.0102 \\ -45.2116 \\ 28.5424 \\ -5.10714 \end{bmatrix} \tag{27}$$

```
> # Step 3: Backward substitution to get x
  x := LinearAlgebra:-BackwardSubstitute(R,d);
```

$$x := \begin{bmatrix} 0.999751 \\ 1.99976 \\ 3.00035 \\ 4.00005 \\ 5.00014 \end{bmatrix} \quad (28)$$

>

>

On the option "fullspan":

```
> A := 1.0*LinearAlgebra:-RandomMatrix(5,3,generator=-9..9);
```

$$A := \begin{bmatrix} -2. & 1. & -2. \\ -4. & -2. & 0. \\ 3. & 4. & 6. \\ -4. & -2. & 3. \\ -1. & -9. & -5. \end{bmatrix} \quad (29)$$

```
> (Q,R) := LinearAlgebra:-QRDecomposition(A);
```

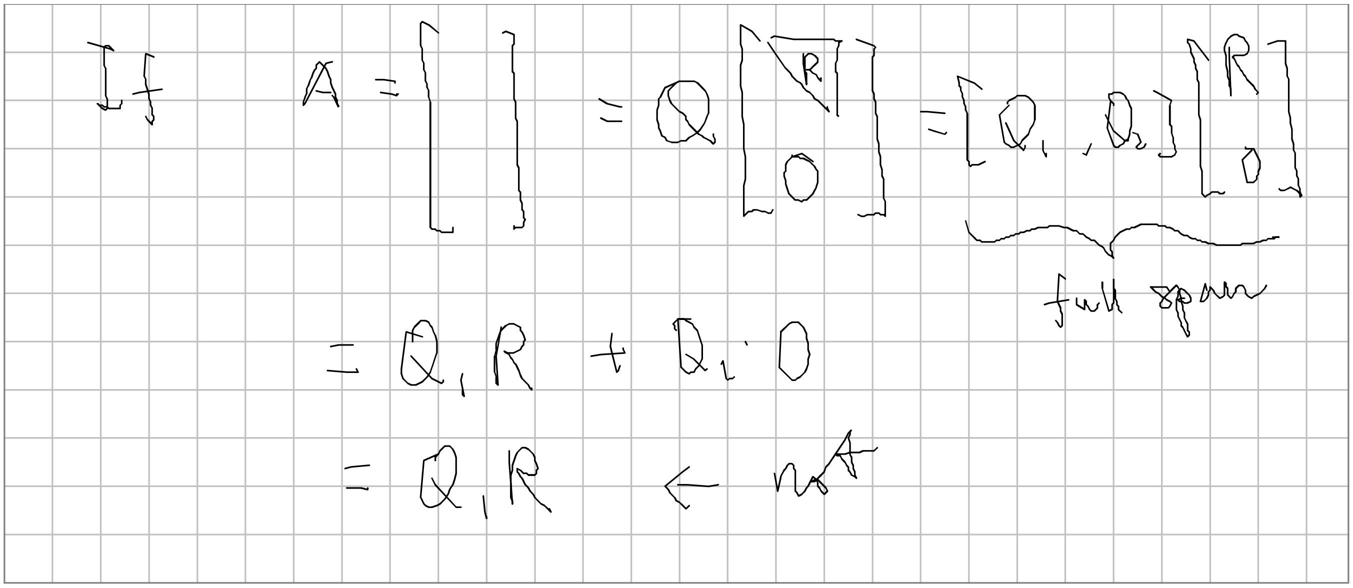
$$Q, R := \begin{bmatrix} -0.29488 & 0.283062 & 0.473416 \\ -0.589761 & 0.117122 & -0.105658 \\ 0.442321 & 0.192775 & -0.639490 \\ -0.589761 & 0.117123 & -0.591508 \\ -0.147440 & -0.924818 & -0.0766940 \end{bmatrix}, \begin{bmatrix} 6.78237 & 5.16041 & 2.21160 \\ 0. & 8.90896 & 5.56598 \\ 0. & 0. & -6.17481 \end{bmatrix} \quad (30)$$

```
> (U,S) := LinearAlgebra:-QRDecomposition(A,fullspan);
```

$$U, S := \begin{bmatrix} -0.29488 & 0.283062 & 0.473416 & -0.435866 & 0.647151 \\ -0.589761 & 0.117122 & -0.105658 & -0.522902 & -0.594873 \\ 0.442321 & 0.192775 & -0.639490 & -0.561690 & 0.206745 \\ -0.589761 & 0.117123 & -0.591508 & 0.386521 & 0.373068 \\ -0.147440 & -0.924818 & -0.0766940 & -0.267767 & 0.213072 \end{bmatrix}, \quad (31)$$

$$\begin{bmatrix} 6.78237 & 5.16041 & 2.21160 \\ 0. & 8.90896 & 5.56598 \\ 0. & 0. & -6.17481 \\ 0. & 0. & 0. \\ 0. & 0. & 0. \end{bmatrix}$$

>



Homework: 1. Generate a 4x2 random matrix, and find Q, R using the Householder transformation

2. Generate a matrix like

```
> A := LinearAlgebra:-RandomMatrix(5,5,generator=-9..9,
  outputoptions=[shape=triangular])[1..-1,1..-1]:
  A[5,1] := 3:
  A;
```

$$\begin{bmatrix} -3 & -4 & -2 & 6 & -2 \\ 0 & 7 & -6 & 3 & 7 \\ 0 & 0 & -7 & 0 & 1 \\ 0 & 0 & 0 & 2 & -6 \\ 3 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \textbf{(32)}$$

Use a sequence of Givens rotation to obtain the QR decomposition.