Note Title 2/24/2009

Examples of L.T.

1. $V = \{$ differentiable functions on $[0,1]\}$

$$T : V \to V$$

where

$$T(f) = f'$$

Proof $\quad T(f+g) = (f+g)' = f' + g'$

$$T(\alpha f) = (\alpha f)' = \alpha \cdot f'$$

QED

2. Same V as above

$S : V \to V$ with $S(f) = f + 1$

This is not a L.T.

Proof $\quad S(2f) = 2f + 1 \neq 2S(f) = 2(f+1)$

3 $\quad V = R_{2,2} \qquad L(A) = A^T$ is a L.T.

Proof $\quad L(A+B) = (A+B)^T = A^T + B^T = L(A) + L(B)$

$$L(\alpha A) = (\alpha A)^T = \alpha A^T = \alpha L(A)$$

QED

4. $V = \mathbb{R}_{2,2}$ $\quad L : \mathbb{R}_{2,2} \to \mathbb{R}_{2,2}$

$$L(A) = \begin{cases} A^{-1} & \text{if } A \text{ is nonsingular} \\ 0 & \text{otherwise} \end{cases}$$

No!

$$L(A+B) = \begin{cases} (A+B)^{-1} \\ 0 \end{cases}$$

$$(A+B)^{-1} \neq A^{-1} + B^{-1} \neq L(A) + L(B)$$

5. $T : \mathbb{R}_1[x] \to \mathbb{R}_2[x]$

with $T(x+1) = x^2 - 1, \quad T(x-1) = x^2 + x$

and $T(\alpha(x+1) + \beta(x-1)) = \alpha(x^2-1) + \beta(x^2+x)$

$\forall \alpha, \beta \in \mathbb{R}$

Proof We 1^{st} prove $(x+1, x-1)$ is a basis for $\mathbb{R}_1[x]$. It suffices to prove it is lin. indep.

Let $\alpha(x+1) + \beta(x-1) = 0$

$\therefore (\alpha+\beta)x + (\alpha-\beta)\cdot 1 = 0$

$\therefore \alpha+\beta = \alpha-\beta = 0 \quad \therefore \alpha = \beta = 0$

Now $\forall f, g \in R_1[z]$

$\exists \alpha, \beta, \alpha', \beta'$ s.t. $f = \alpha(z+1) + \beta(z-1)$

$g = \alpha'(z+1) + \beta'(z-1)$

$\therefore f + g = \underbrace{(\alpha+\alpha')(z+1) + (\beta+\beta')(z-1)}$

$\therefore T(f+g) = T(\qquad\qquad)$

$= (\alpha+\alpha')(z^2-1) + (\beta+\beta')(z^2+z)$

$= [\alpha(z^2-1) + \beta(z^2+z)] + [\alpha'(z^2-1) + \beta'(z^2+z)]$

$= \quad T(f) \qquad\qquad + \quad T(g)$

Similarly $T(\gamma f) = \gamma T(f)$ QED

6. Let $L : V \to W$ be linear be defined by

Prove $L(0_V) = 0_W$

$0 = 0 + 0$

Proof $L(0_V) = L(0_V + 0_V)$ (A2)

$\therefore L(0_V) = L(0_V) + L(0_V)$ (L.T)

$L(0_V) - L(0_V) = (L(0_V) + L(0_V)) - L(0_V)$ (A3)

$0_W = L(0_V)$

QED

7. Let $L: V \to W$ be linear

Prove NullSpace (L) is a V.S.

Proof. $\forall\, u, v \in$ NullSpace (L)

$\alpha, \beta \in F$

$$L(\alpha u + \beta v) = L(\alpha u) + L(\beta v)$$
$$= \alpha L(u) + \beta L(v)$$
$$= \alpha \cdot 0 + \beta \cdot 0$$
$$= 0$$

$\therefore$ $\alpha u + \beta v \in$ NullSpace (L) QED

HW. download proof problems §5.1

Select 8 problems

Extra credit for more than 8.