Note Title 1/20/2009

P 23 #19

Proof. For $k=1$ $A^{k+1} = A^2 = (B+C)^2 = (B+C)(B+C)$

$= B^2 + BC + CB + C^2$

$= B^2 + BC + BC + 0$

$= B^2 + 2BC = B(B+2C) = B^k(B+(k+1)C)$

Assume the conclusion is true for $k=n$

Namely $A^{n+1} = B^n(B+(n+1)C)$

For $k=n+1$, $A^{k+1} = A^{n+1+1} = A^{n+1} \cdot A = B^n(B+(n+1)C)\cdot(B+C)$

$= B^n(B^2 + BC + (n+1)CB + (n+1)C^2)$

$= B^n(B^2 + BC + (n+1)BC) = B^n(B^2 + (n+2)BC)$

$= B^{n+1}(B+(n+2)C)$ QED

#8 p29 $S^3 = 0 \Rightarrow (I-S)^{-1} = I + S + S^2$

Proof

$(I-S)(I+S+S^2)$

$= I + S + S^2 - S(I+S+S^2)$

$= I + S + S^2 - S - S^2 - S^3$

$= I + 0 + 0 + 0$

$= I$ QED

$A^{-1} = B$

$\Leftrightarrow A \cdot B = I$

#5 p40 :

If $AB = BA$, B^{-1} exists then $AB^{-1} = B^{-1}A$

Proof $\because AB = BA$ $\therefore B^{-1}AB = B^{-1}BA = A$

$\therefore B^{-1}AB \cdot B^{-1} = AB^{-1}$

$\therefore B^{-1}A = AB^{-1}$ QED

#11 p29. If $AB = BA$, and A^{-1}, B^{-1} exist.
Then $A^{-1}B^{-1} = B^{-1}A^{-1}$

Proof $$A^{-1}B^{-1} = (BA)^{-1} = (AB)^{-1} = B^{-1}A^{-1}$$ QED

Thm. $(AB)^{-1} = B^{-1}A^{-1}$ $\qquad (AB)^T = B^T A^T$

#6 p29

Proof
$$\frac{1}{9} A \cdot A = \frac{1}{9}\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ QED

Example Prove $(A+BB^T)^{-1}B = A^{-1}B(I+B^TA^{-1}B)^{-1}$

Proof. $B(I+B^TA^{-1}B) = B + BB^TA^{-1}B$

$$(A+BB^T)A^{-1}B = AA^{-1}B + BB^TA^{-1}B$$
$$= B + BB^TA^{-1}B$$

$\therefore$ $B(I+B^TA^{-1}B) = (A+BB^T)A^{-1}B$

$\therefore$ $(A+BB^T)^{-1}B(I+\cancel{B^TA^{-1}}B) \cdot (I\cancel{+B^T}A^{-1}B)^{-1}$

$$= (A+\cancel{BB^T})^{-1}(A\cancel{+BB^T})A^{-1}B(I+B^TA^{-1}B)^{-1}$$

QED.

Example $\quad (I+BA)^{-1}B = B(I+AB)^{-1}$

Proof. $\quad B(I+AB) = B + BAB$

$$(I+BA)B = B + BAB$$

$$\therefore \quad B(I+AB) = (I+BA)B$$

$$(I+BA)^{-1}B\cancel{(I+AB)}\cancel{(I+AB)^{-1}}$$

$$= \cancel{(I+BA)^{-1}}\cancel{(I+BA)}B(I+AB)^{-1}$$

$$\therefore \quad (I+BA)^{-1}B = B(I+AB)^{-1}$$

Example. Prove $(A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B$

Proof $\quad LHS = [B^{-1}(BA^{-1}+I)]^{-1}$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

$$= [B^{-1}(BA^{-1}+AA^{-1})]^{-1}$$

$$= [B^{-1}(B+A)A^{-1}]^{-1}$$

$$= (A^{-1})^{-1}(B+A)^{-1}(B^{-1})^{-1} = RHS$$

QED

HW : Prove

(1) $(I+A^{-1})^{-1} = A(A+I)^{-1}$

(2) $A^{-1}+B^{-1} = A^{-1}(A+B)B^{-1}$

(3) If $B^TB = I$ then $(BB^T)^2 = BB^T$

(4) ~~A~~ If A and B are invertible so are AB^{-1} and BA^{-1}

Extra credit Prove

$$(I+AB)^{-1} = I - A(I+BA)^{-1}B$$