Answer to proof problems for §5.1

Note Title 3/3/2009

1. Range(L) is a vector space, where $L: V \to W$ is linear

Proof. $\forall u, v \in$ Range(L) and $\alpha, \beta \in F$

$\exists x, y \in V$ s.t. $u = L(x)$, $v = L(y)$

$\therefore \alpha u + \beta v = \alpha L(x) + \beta L(y) = L(\alpha x) + L(\beta y)$

$= L(\alpha x + \beta y)$

$\therefore \alpha u + \beta v \in$ Range(L) $\therefore$ It is a vector space QED

2. $L: V \to W$ is linear $\Rightarrow L(u-v) = L(u) - L(v)$

Proof. (We'll use Exercise 1(b), page 99)

$L(u-v) = L(u+(-v))$ (Definition of $u-v$)

$= L(u+(-1)v))$ (Exercise 1(b), p99)

$= L(u) + (-1)L(v)$ (linearity)

$= L(u) + (-L(v))$ (Exercise 1(b), p99)

$= L(u) - L(v)$ QED

3. Let $L: V \to W$ be linear. Prove $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$

$\forall \alpha, \beta \in F$, $\forall u, v \in V$.

Proof. $L(\alpha u + \beta v) = L(\alpha u) + L(\beta v)$

$= \alpha L(u) + \beta L(v)$ QED

4. Let $L: V \to W$. If $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$ holds for all $u, v \in V$, $\alpha, \beta \in F$. Then L is linear

Proof. Let $\alpha = \beta = 1$. Then $L(u+v) = L(u) + L(v)$

Let $\beta = 0$ Then

$L(\alpha u) = L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) = \alpha L(u)$

QED

5. Let $L: V \to W$ be linear. Prove

$$L(\alpha_1 u_1 + \cdots + \alpha_n u_n) = \alpha_1 L(u_1) + \cdots + \alpha_n L(u_n)$$

for all $u_1, \cdots, u_n \in V$, $\alpha_1, \cdots, \alpha_n \in F$.

Proof. For $n=1$. $L(\alpha_1 u_1) = \alpha_1 L(u_1)$ by Def 5.1.1 (2)

Assume the statement is true for $n=k$. Namely

$$L(\alpha_1 u_1 + \cdots + \alpha_k u_k) = \alpha_1 L(u_1) + \cdots + \alpha_k L(u_k)$$

Then for $n = k+1$

$$\begin{aligned} L(\alpha_1 u_1 + \cdots + \alpha_k u_k + \alpha_{k+1} u_{k+1}) &= L(\alpha_1 u_1 + \cdots + \alpha_k u_k) + L(\alpha_{k+1} u_{k+1}) \\ &= (\alpha_1 L(u_1) + \cdots + \alpha_k L(u_k)) + \alpha_{k+1} L(u_{k+1}) \\ &= \alpha_1 L(u_1) + \cdots + \alpha_k L(u_k) + \alpha_{k+1} L(u_{k+1}) \end{aligned}$$

$\therefore$ The statement is true for $n = k+1$. QED.

6 Prove $T(v) = 0$ is a linear transformation

Proof $T(u+v) = 0 = 0+0$

$= T(u) + T(v)$

$T(\alpha u) = 0 = \alpha \cdot 0 = \alpha T(u)$

$\therefore$ T is a L.T. QED

7. Prove $T(v) = v$, $\forall v \in V$ is a L.T.

Proof. $\forall u, v \in V$. $T(u+v) = u+v = T(u) + T(v)$

$T(\alpha u) = \alpha u = \alpha T(u)$

$\therefore$ T is a L.T. QED.

8 Let $T: V \to W$, $S: V \to W$ be linear. Assume

$V = \langle v_1, \cdots, v_n \rangle$ and $T(v_i) = S(v_i)$ $\forall i$

Then $T(u) = S(u)$ $\forall u \in V$.

$\because\ V = \langle v_1, \cdots, v_n \rangle$

Proof. $\because\ \forall\ u \in V,\ \exists\ \alpha_1, \cdots, \alpha_n$ s.t. $u = \alpha_1 v_1 + \cdots + \alpha_n v_n$

$\therefore\ T(u) = T(\alpha_1 v_1 + \cdots + \alpha_n v_n)$

$= \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$ (Problem 8)

$= \alpha_1 S(v_1) + \cdots + \alpha_n S(v_n)$ $(T(v_i) = S(v_i))$

$= S(\alpha_1 v_1 + \cdots + \alpha_n v_n)$ (Problem 8)

$= S(u)$ QED

9 Let $L: V \to W$ be linear. If $L(u_1), \cdots, L(u_n)$ are li. indep. So are $u_1, \cdots, u_n$.

Proof. Let $\alpha u_1 + \cdots + \alpha_n u_n = 0$

Then $L(\alpha u_1 + \cdots + \alpha u_n) = L(0) = 0$

$\therefore\ 0 = \alpha_1 L(u_1) + \cdots + \alpha_n L(u_n)$

$\because\ L(u_1), \cdots, L(u_n)$ are li. indep.

$\therefore\ \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

$\therefore\ u_1, \cdots, u_n$ are li. indep.

10 Let $L: V \to W$ be linear. If $u_1, \cdots, u_n$ are li. dep in V Then $L(u_1), \cdots, L(u_n)$ are li. dep.

Proof. $\because\ u_1, \cdots, u_n$ are li. dep.

$\therefore\ \exists\ \alpha_1, \cdots, \alpha_n$, not all zero, s.t. $\alpha_1 u_1 + \cdots + \alpha_n u_n = 0$

$\therefore\ \alpha L(u_1) + \cdots + \alpha_n L(u_n) = L(\alpha_1 u_1 + \cdots + \alpha_n u_n) = L(0) = 0$

$\therefore\ L(u_1), \cdots, L(u_n)$ are li. dep. QED