About homework

1. $$A_k - sI = Q_k R_k$$

$$A_{k+1} = R_k Q_k + sI$$

A_{k+1} has the same eigenvalues as A_k

Proof. $$R_k = Q_k^T (A_k - sI) = Q_k^T A_k - s Q_k^T$$

$$\therefore \quad A_{k+1} = (Q_k^T A_k - s Q_k^T) Q_k + sI$$

$$= Q_k^T A_k Q_k - sI + sI = Q_k^T A_k Q_k$$

$\therefore$ A_{k+1} has the same eigenvalues a A_k

2. $$P(x) = 3x^6 - 5x^3 + 3x^2 - 8 = 3\left(x^6 - \tfrac{5}{3}x^3 + x^2 - \tfrac{8}{3}\right)$$

has the same roots as

$$x^6 + 0 \cdot x^5 + 0x^4 - \tfrac{5}{3}x^3 + x^2 - \tfrac{8}{3}$$

$+0 \cdot x$

The companion matrix

$$\begin{bmatrix} 0 & & & & & +8/3 \\ 1 & \ddots & & & & 0 \\ & \ddots & \ddots & & & -1 \\ & & \ddots & \ddots & & 5/3 \\ & & & \ddots & \ddots & 0 \\ & & & & 1 & 0 \end{bmatrix}$$

its eigenvalues are roots we want.

Chaper 8. Differential equations. (overview)

~~ording~~ (ODE)

The best method is Runge-Kutta method

(see Maple demo)

Problem $\begin{cases} \dfrac{dy}{dt} = f(t,y) & t \in [a,b] \\ y(a) = y_0 \end{cases}$

Objective : a function $y = y(t)$

For numerical computation : a table

t	t_1	t_2	$\cdots$	t_n
y	y_1	y_2	$\cdots$	y_n

$y_k \approx y(t_k)$

Example 8.1.3 p333

$$\frac{dy}{dt} = y^2 . \qquad t \in [0,2]$$

$$y(0) = 1$$

Page 356. #3. $y' = x^2 - y^2$, $y(0) = 1$, $[0,1]$

$$\begin{cases} \dfrac{dy}{dt} = t^2 - y^2 & t \in [0,1] \\ y(0) = 1 \end{cases}$$

Page 357 $y''' = e^x y$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$

Let $u_1 = y$, $u_2 = y'$, $u_3 = y''$, $t = x$

$$\begin{cases} u_1' = u_2 \\ u_2' = u_3 & t \in [0,2] \\ u_3' = e^t u_1 \\ u_1(0) = 1, \quad u_2(0) = 0, \quad u_3(0) = 1 \end{cases}$$

$$\begin{cases} \vec{u}' = \vec{f}(t, \vec{u}) \\ \vec{u}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

For solving a (square) system $F(\vec{x}) = 0$, we can use Newton-Raphson's iteration

$$\vec{x}_{k+1} = \vec{x}_k - J(\vec{x}_k)^{-1} F(\vec{x}_k)$$

square system : # of equation = # of variables

How about a non-square system? For example : If we have a model

$$y = c - a e^{-bt}$$

with data

x	1.2	2.0	3.2	0
y	2.9	2.982	2.998	2

Thus, we have the following system :

$$\begin{cases} c - a e^{-b\cdot(1.2)} - 2.9 = 0 \\ c - a e^{-b\cdot(2.0)} - 2.982 = 0 \\ c - a e^{-b(3.2)} - 2.998 = 0 \\ c - a e^{-b\,0} - 2 = 0 \end{cases} \qquad \vec{F}(a,b,c) = 0 \qquad 4 \times 3$$

Gauss-Newton iteration

$$\vec{x}_{k+1} = \vec{x}_k - J(\vec{x}_k)^{+} F(\vec{x}_k)$$

Here, for any matrix A, $A^{+} = (A^T A)^{-1} A^T$ is called the pseudo-inverse

Question ; How to carry out Gauss Newton iteration without computing $J(\vec{x}_k)^{+}$

Hint : (1) See how to do N-R method without $J(\vec{x}_k)^{-1}$

(2) When we solve LS problem $Ax = b$,

$$x = A^{+} b = (A^T A)^{-1} A^T b \quad \text{(in theory)}$$

You may want to try the above problem with initial guess $a = 0.99$, $b = 2.01$, $c = 2.99$