Standard iterations (Newton-Raphson, etc) can't get accurate results for "multiple zeros"

Example : $f(x) = (x-1)^3 (x+5) = (x-1)(x-1)(x-1)(x+5)$

$x=1$ is triple zero (multiplicity $= 3$)

If we use 10-digits computation the "attainable" accuracy

$$\approx \frac{10}{3} \approx 3 \text{ digits accuracy}$$

on the zero $x=1$

$$\frac{\text{machine precision}}{\text{multiplicity}}$$

§ 6.9 Systems of Nonlinear equations

Example $\begin{cases} x^2 - y^2 + 2y = 0 \\ 2x + y^2 - 6 = 0 \end{cases} \Rightarrow F(z) = 0$

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad F(z) = \begin{bmatrix} x^2 - y^2 - 2y \\ 2x + y^2 - 6 \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix}$$

Recall Newton-Raphson :

$$x_{k+1} = x_k - (f'(x_k))^{-1} f(x_k)$$

for single variable $f(x)$

The multivariate version of $f'(x_k)$ is the Jacobian :

$$J(z) = \begin{array}{c} \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} \begin{array}{l} \leftarrow F_1 \\ \\ \leftarrow F_2 \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ x & y \end{array} \end{array}$$

The Newton - Raphson method

$$z_{k+1} = z_k - [J(z_k)]^{-1} F(z_k) \qquad k = 0, 1, 2, \dots$$

The actual process is based on

$$J(z_k)(z_{k+1} - z_k) = -F(z_k)$$

$$\begin{cases} J(z_k) \cdot u_k = -F(z_k) & \text{solve for } u_k \\ z_{k+1} = z_k + u_k \end{cases}$$

As usual, we need a good initial guess

Homework #3

plot the graph of

$$\begin{cases} x^4 + y^4 - 1 = 0 \\ x^5 y^2 - 4x^3 y^3 + x^2 y^5 - 1 = 0 \end{cases}$$

in $-2 \le x \le 2, \quad -2 \le y \le 2$

and use Newton - Raphson's iteration to find two solutions